Abstract
Many transformation systems for program optimization, program synthesis, and program specialization are based on fold/unfold transformations. In this paper, we present a fold/unfold-based transformation framework for rewriting logic theories which is based on narrowing. For the best of our knowledge, this is the first fold/unfold transformation framework which allows one to deal with functions, rules, equations, sorts, and algebraic laws (such as commutativity and associativity). We provide correctness results for the transformation system w.r.t. the semantics of ground reducts. Moreover, we show how our transformation technique can be naturally applied to implement a Code Carrying Theory (CCT) system. CCT is an approach for securing delivery of code from a producer to a consumer where only a certificate (usually in the form of assertions and proofs) is transmitted from the producer to the consumer who can check its validity and then extract executable code from it. Within our framework, the certificate consists of a sequence of transformation steps which can be applied to a given consumer specification in order to automatically synthesize safe code in agreement with the original requirements. We also provide an implementation of the program transformation framework in the high-performance, rewriting logic language Maude which, by means of an experimental evaluation of the system, highlights the potentiality of our approach.

Categories and Subject Descriptors L.2.2 [Artificial Intelligence]: Automatic Programming—Program Transformation; G.4 [Mathematics of Computing]: Mathematical Software—Certification and testing

General Terms Languages, Performance

Keywords Rewriting Logic, Fold/Unfold Program Transformation, Code Carrying Theory

1. Introduction
Transforming programs automatically to optimize their efficiency is one of the most fascinating techniques for rule–based programming languages [33, 43]. One of the most extensively studied program transformation approaches is the so called fold/unfold transformation system [8, 9, 16] (also known as the rules+strategies approach [35]). In this approach, the goal of obtaining a correct and efficient program is achieved in two phases, which may be performed by different actors: the first phase consists in writing an initial, maybe inefficient, program whose correctness can be easily shown; the second phase consists in transforming the initial program in order to obtain a more efficient one. This is often done by constructing a sequence of equivalent programs —called transformation sequence and usually denoted by \( R_0, \ldots, R_n \)— where each program \( R_i \) is obtained from the preceding ones \( R_0, \ldots, R_{i-1} \) by using an elementary transformation rule. The essential rules are folding and unfolding, i.e., contraction and expansion of subexpressions of a program using the definitions of the program itself (or of a preceding one). Other rules which have been considered are, e.g., instantiation, definition introduction/elimination and abstraction (sometimes referred with different names). When performing program transformation we may end up with a final program which is equal to the initial one, since the folding rule is the inverse of the unfolding rule. Thus, during the transformation process, we need strategies which guide the application of the transformation rules and can allow one to derive programs with improved performance. Some popular transformation strategies which have been proposed in the literature are the composition and tupling strategies. The composition strategy [9] is used to avoid the construction of intermediate data structures that are produced by some function \( g \) and consumed by another function \( f \). For some class of programs the composition strategy can be applied automatically. The tupling strategy [9, 17] proceeds by grouping calls with common arguments together so that their results are computed simultaneously. Unfortunately, the tupling strategy is more involved than the composition strategy and can in general be obtained only semi-automatically (although for particular classes of programs the tupling strategy has been completely automated [11, 12]).

A lot of literature has been devoted to proving the correctness of fold/unfold systems w.r.t. the various semantics proposed for logic programs [6, 22, 25, 27, 34, 38], functional programs [36, 37], and functional logic programs [1]. Quite often, however, transformations may have to be carried out in contexts in which the function symbols satisfy certain equational axioms. For example, in rule-based languages such as ASF+SDF [4], Elan [5], OBJ [23], CafeOBJ [18], and Maude [15] some function symbols may be de-
clared to obey given algebraic laws (the so-called \textit{equational attributes} of OBJ, CafeOBJ and Maude), whose effect is to compute with equivalence classes modulo such axioms while avoiding the risk of non-termination. Similarly, theorem provers, both general first-order logic ones and inductive theorem provers, routinely support commonly occurring equational theories for some function symbols such as associativity-commutativity. Moreover, several of the above-mentioned languages and provers have an expressive \textit{order-sorted} typed setting with sorts and subsorts (where subsort inclusions form a partial order and are interpreted semantically as set-theoretic inclusions of the corresponding data sets). Surprisingly, to the best of our knowledge there seems to be no program transformation framework coping with sorts, rules, equational theories, and algebraic laws. In this paper, we develop our fold/unfold–based transformation methodology in the framework of rewriting logic \cite{29}, a flexible and expressive \textit{logical framework} in which a wide range of logics and models of computation can be faithfully represented. Besides sorts, rules, equations and algebraic laws (such as commutativity and associativity), rewriting logic also features a powerful tool for unifying (a combination of term rewriting and unification \cite{21}), and is efficiently implemented in the functional programming language Maude \cite{13, 15}.

\textbf{Our contribution.} The main contributions of the paper can be summarized as follows:

- We propose the first fold/unfold framework in the literature that applies to rewriting logic theories \cite{29} and prove its correctness. Our methodology considers the possibility of transforming the equation set and the rule set of a rewrite theory separately in a way the semantics of ground reducts is proved to be preserved. We employ narrowing in order to empower the unfold operation by calculating the instance of an existing rule to embed the unfolding rule automatically via unification. The auxiliary transformation rules adopted apart from fold and unfold are: definition introduction and elimination, and abstraction.

- We chose to apply our transformation framework to the problem of securing the transfer of code from a code producer to a code consumer. Among the many different solutions that have been proposed to tackle this problem, we adhere to Code Carrying Theory (CCT) \cite{40, 41} — a program synthesis framework stemming from a pioneering work of Manna and Waldinger \cite{28} in which a theorem proving approach is taken to synthesize correct code from theorems and proofs induced by user specifications. As opposed to more traditional certification approaches such as proof–carrying code \cite{32}, where both the code and the certificate are transmitted from the code producer to the code consumer, in CCT only the certificate is transmitted in the form of a theory (a set of axioms and theorems) together with a set of proofs of the theorems; no code needs to be explicitly transmitted. The code consumer would admit new axioms from the code producer only if the associated proofs actually do prove the theorems. If this checking succeeds, then the code consumer can apply a code extractor to the set of function-defining axioms obtained to produce the executable code. The form of the function-defining axioms is such that it is easy to extract executable code from them. By using the proposed system fold/unfold transformations, which can be applied to a wide class of programs automatically, our CCT methodology greatly reduces the burden on the code producer.

The key idea behind our CCT methodology is as follows. Assuming the code consumer provides the requirements in the form of a rewrite theory, the code producer can (semi-) automatically obtain an efficient implementation of the specified functions by applying a sequence of transformation rules. Moreover, having proved the correctness of the transformation system, the code producer can transmit as the required certificate just a compact representation of the sequence of transformation rules to the consumer so he does not need to manually construct any other correctness proof. By applying the transformation rules to the initial requirements, the code consumer can inexpensively obtain the executable code that can be eventually compiled to a different target language if needed.

- We provided an implementation of our transformation methodology, and we conducted an experimental evaluation of the system, which demonstrates the usefulness of our approach.

\textbf{Related work.} In \cite{40, 41}, an implementation of CCT has been presented using ATHENA \cite{2, 3}, which is a tool that provides a language for both ordinary computation and for logical deduction. To the best of our knowledge, no other implementation of CCT has been proposed in the related literature. The system presented in \cite{41} requires manually defining a set of axioms, which are the basis for providing an efficient implementation of the specified functions; then, a proof of the correctness conditions required by the consumer is manually constructed using the previously defined axioms.

A different approach to program transformation is proposed in \cite{10}, where a term–rewriting transformation framework is formalized by using templates. In this approach, programs are expressed as Term Rewriting Systems (TRSs) \cite{26}, and are transformed according to a given program transformation template expressed as a TRS too. Such a template consists of program schemas for input and output programs and a set of equations that the input and output programs must validate to guarantee the correctness of the transformation. A library of templates that matches the structure of the programs is required, otherwise the transformation cannot be applied.

\textbf{Plan of the paper.} The rest of the paper is organized as follows. In Section 2, we recall some necessary notions about rewriting logic and the Maude language. Section 3 formalizes the fold/unfold transformation system for rewrite theories and proves its correctness. In Section 4, we describe how the program transformation system can be exploited to implement CCT. Section 5 presents the prototypical implementation of our transformation framework. In Section 6, we draw some conclusions and discuss future work. The proofs of all the results of the paper are given in Appendix A. Finally, Appendix B outlines a technique for obtaining coherent and consistent equational theories.

\section{Preliminaries}

We consider an \textit{order-sorted} signature $\Sigma$, with a finite poset of sorts $(S, \leq)$. We assume an $S$-sorted family $X = \{X_s\}_{s \in S}$ of disjoint variable sets, $T_S(X)$, and $T_S$ are the sets of terms and ground terms of sort $s$, respectively. We write $T_S(X)$ and $T_S$ for the corresponding term algebras. The set of variables occurring in a term $t$ is denoted by $\text{Var}(t)$. We write $\sigma t$ for the list of syntactic objects $o_1, \ldots, o_n$.

A \textit{position} $p$ in a term $t$ is represented by a sequence of natural numbers ($\Lambda$ denotes the empty sequence, i.e., the root position). Positions are ordered by the \textit{prefix} ordering: $p \leq q$, if $\exists w$ such that $p \cdot w = q$. Given a term $t$, we let $\text{Pos}(t)$ and $\text{NPos}(t)$ respectively denote the set of positions and the set of non-variable positions of $t$ (i.e., positions where a variable does not occur). $t[p]$ denotes the \textit{subterm} of $t$ at position $p$, and $t[p][s]$ denotes the result of \textit{replacing} the subterm $t[p]$ by the term $s$.

A substitution $\sigma$ is a mapping from variables to terms $\{x_1/t_1, \ldots, x_n/t_n\}$ such that $x_\sigma = t_i$ for $i = 1, \ldots, n$ (with $x_i \neq x_j$ if $i \neq j$), and $x \sigma = x$ for all other variables $x$.

An \textit{(order-sorted) equational theory} is a pair $\mathcal{E} = (\Sigma, \Delta \cup B)$, where $\Sigma$ is an order-sorted signature, $\Delta$ is a collection of equations ($l = r$), and $B$ is a collection of equational axioms for
associativity, commutativity, and identity declared for the different defined functions. We assume $\Sigma$ can be always considered as the disjoint union $\Sigma = C \sqcup D$ of symbols $c \in C$, called constructors, and symbols $f \in D$, called defined functions, each one having a fixed arity, where $D = \{ f \mid f(T) = r \in \Delta \}$ and $C = \Sigma - D$. Then $T(C, \Delta)$ is the set of constructor terms. Given an equation $l = r$, terms $l$ and $r$ are called the left-hand side (or lhs) and the right-hand side (or rhs) of the equation, respectively, and $\text{Var}(r) \subseteq \text{Var}(l)$.

The equations in an equational $\Sigma$ are considered as simplification rules by using them only in the left to right direction, so for any term $t$, by repeatedly applying the equations as simplification rules, we eventually reach a term to which no further equations apply. The result is called the canonical form of $t$ w.r.t. $E$. This is guaranteed by the fact that $E$ is required to be terminating and Church-Rosser [7]. The set of equations in $\Delta$ together with the equational axioms of $B$ in an equational theory $E$ induce a congruence relation on the set of terms $T_{\Sigma}(X)$ which is usually denoted by $\equiv_{E}$. $E$ is a presentation or axiomatization of $\equiv_{E}$. In abuse of notation, we speak of the equational theory $E$ to denote the theory axiomatized by $E$. Given an equational theory $E$, we say that a substitution $\sigma$ is an $E$-unifier of two generic terms $t$ and $t'$ if $\sigma t = \sigma t'$. We define the one-step rewrite relation $(\text{order-sorted})$ rewrite theory as

$$(1) \text{ Non-erasing, } \text{ if } \forall \text{Var}(v) = \forall \text{Var}(t').$$

$$(2) \text{ Sort preserving, if for each substitution } \sigma, \text{ we have } t \sigma \in T_{\Sigma}(X)_s \text{ if and only if } t' \sigma \in T_{\Sigma}(X)_s.$$ 

$$(3) \text{ Sort decreasing, if for each substitution } \sigma, \text{ } t' \sigma \in T_{\Sigma}(X)_s \text{ implies } t \sigma \in T_{\Sigma}(X)_s.$$ 

$$(4) \text{ Left (or right) linear, if } t \text{ (resp. } t') \text{ is linear, i.e., no variable occurs in the term more than once. It is called linear if both } t \text{ and } t' \text{ are linear.}$$

A set of equations/rules is said to be non-erasing, or sort decreasing, or sort preserving, or (left or right) linear, if each equation/rule in it is so. Two equations of the form $l_0 = r_0$ and $l_1 = r_1$ overlap if there is a position $p_0 \in N \text{V}P \text{os}(l_0)$ and substitution such that $l_0[p_0] = t_0 l_1 \sigma$, or, viceversa, a position $p_1 \in N \text{V}P \text{os}(l_1)$ and substitution $\sigma$ such that $l_1[p_1] = t_0 l_1 \sigma$. A set of equations is said to be non overlapping if there is no pair of overlapping rules. The same property can be easily lifted to rewrite rules.

We define the one-step rewrite relation on $T_{\Sigma}(X)$ as follows: $t \rightarrow_R t'$ if there is a position $p \in P \text{os}(l)$, a rule $l \rightarrow_r r$, and a substitution $\sigma$ such that $l[p] = l \sigma$ and $t' = t[\sigma]$. An instance $l \sigma$ of a rule $l \rightarrow_r r$ is called a redex. A term without redexes is called normal form. A rewrite theory $R$ is weakly normalizing if every term $t$ has a normal form in $R$, though infinite rewrite sequences from $t$ may exist. The relation $\rightarrow_{R/E}$ for rewriting modulo $E$ is defined as $\equiv_{E} \circ \rightarrow_R \circ \equiv_{E}$. Let $\subseteq A \times A$ be a binary relation on a set $A$. We denote the transitive closure by $\rightarrow^\ast$, the reflexive and transitive closure by $\rightarrow$, and the rewrite up to normal form by $\rightarrow^\ast$. A rewrite theory is sufficiently complete if enough rules/equations have been specified, so that functions of the theory are fully defined on all relevant data. In the order sorted context, due to the sub-sort relation and overloading, sufficient completeness for weakly-normalizing, confluent and sort decreasing rewrite theories, can be checked by showing that for each term $f(t_1, \ldots, t_n)$ where $f : s_1 \ldots s_n \rightarrow s$ is a defined symbol and every $t_i$ is a constructor term of sort $s_i$, $f(t_1, \ldots, t_n)$ is reducible in $R \cup E$ [24].

**Example 3.1.** Consider the following rewrite theory $\{ \Sigma, \Delta \}$

such that $C_1 = \{ b, e \}$, $C_2 = \{ b, c, d \}$, $D_1 = \{ a, d \}$, $D_2 = \{ f \}$, $\Delta = \{ a = b, b = d \}$, $R = \{ f(b, c) \rightarrow d \}$ where $B$ contains the commutativity axiom for $f$. Then we can $R/E$ rewrite $f(c, a) \rightarrow f(e, b) = B f(b, c) \rightarrow_R d = \Delta e$.

**3. Transforming Rewrite Theories**

In this section, we introduce the narrowing-based transformation rules over rewrite theories and establish the correctness of the transformation system. We divide the transformation process into two steps. At the first step, we disregard the rewrite rules and we only transform the set of equations $\Delta$ of the equational theory modulo the set of equational axioms $B$ (which are left unchanged). Then, we consider a new rewrite theory made of the transformed equational theory and the original rewrite rules. At the second step, we transform the rules modulo the new equational theory. This two-step process allows one to transform the rewrite rules modulo a fixed, already optimized, equational theory, which cannot change during the transformation of the rewrite rules. This fact results to be particularly helpful in proving the soundness of the whole fold/unfold framework.

**3.1 Narrowing in Rewriting Logic**

Considering the rewrite relation $\rightarrow_{R/E}$, since $E$-congruence classes can be infinite, $\rightarrow_{R/E}$- reducibility is undecidable in general. One way to overcome this problem is to implement $R/E$-rewriting by a combination of rewriting using ordered equations and rules [42]. We adopt this approach in this paper. We assume the following properties on $E = \Delta \cup B$.

(i) $B$ is non-erasing, and sort preserving.

(ii) $B$ has a finitary and complete unification algorithm, which implies that $B$-matching is decidable, and $\Delta \cup B$ has a complete (but not necessarily finite) unification algorithm.

(iii) $\Delta$ is sort decreasing, and confluent and terminating modulo $B$.

We define the relation $\rightarrow_{\Delta, B}$ on $T_{\Sigma}(X)$ as follows: $t \rightarrow_{\Delta, B} t'$ if there is a position $p \in P \text{os}(l), t \rightarrow_R t'$ and a substitution $\sigma$ such that $l[p] = l \sigma$ and $t'[p] = t'[\sigma]$. The relation $\rightarrow_{R, B}$ is similarly defined, and we define $\rightarrow_{R, \Delta, B} = \rightarrow_{R, B} \cup \rightarrow_{\Delta, B}$. The idea is to implement $\rightarrow_{R/E}$ using $\rightarrow_{R, \Delta, B}$. For this approach to be correct and complete, we need the following additional assumptions [30].

(iv) $\rightarrow_{\Delta, B}$ is coherent with $B$, i.e., $\forall t_1, t_2, t_3$, we have that $t_1 \rightarrow_{\Delta, B} t_2$ and that $t_1 =_{B} t_3$ implies $\exists t_4, t_5$ such that $t_2 \rightarrow_{\Delta, B} t_4$, $t_3 \rightarrow_{\Delta, B} t_5$, and $t_4 =_{B} t_5$.

(v) $\rightarrow_{R, B}$ is $E$-consistent with $B$, i.e., $\forall t_1, t_2, t_3$, we have that $t_1 \rightarrow_{R, B} t_2$ and that $t_1 =_{B} t_3$ implies $\exists t_4$ such that $t_3 \rightarrow_{R, B} t_4$, and $t_2 =_{E} t_4$.
The set of new equations (\(3\)) Every defined function symbol occurring in

\[ t_1 \rightarrow_{R,B} t_2 \]

and that \( t_1 \rightarrow_{\Delta} t_3 \) implies \( \exists t_4, t_5 \) such that

\[ t_3 \rightarrow_{\Delta} t_4, t_4 \rightarrow_{R,B} t_5, \text{and } t_5 =_{E} t_2. \]

Narrowing [21] generalizes term rewriting by allowing free variables

\[ \rightarrow_{\Delta, B} \]

in terms (as in logic programming) and by performing unification (at non-variable positions) instead of matching in order to

\( (\text{non-deterministically}) \) reduce a term. The narrowing relation for rewriting logic theories is defined as follows [30].

**Definition 3.1** \((R \cup \Delta, B, \text{Narrowing})\). Let \( R = (\Sigma, \Delta, \cup, B, R) \) be an ordered-sorted rewrite theory satisfying properties (i) - (vi) above. The \( R \cup \Delta, B\)-narrowing relation on \( T_\Sigma(\mathcal{X}) \) is defined as \( t \sim_{\sigma, R, \cup, \Delta, B} t' \) if there exist \( p \in N \forall \mathcal{Pos}(t) \), a rule \( l \rightarrow r \) or equation \( l = r \) in \( R \cup \Delta, \) and \( \sigma \), which is a \( B\)-unifier of \( t_1 \) and \( l \) such that \( t' = (t[r]_p)\sigma \). \( t \sim_{\sigma, R, \cup, \Delta, B} t' \) is also called a \( R \cup \Delta, B\)-narrowing step.

Moreover, to ensure narrowing completeness we assume that (vii) \( B \) is linear and (viii) \( R \) is sort decreasing and right-linear. Properties (i) - (viii) are enforced on rewrite theories to guarantee the strong reachability-completeness of the narrowing relation \( \sim_{\sigma, R, \cup, \Delta, B} \). Roughly speaking, for each term \( f \) and substitution \( \theta \) such that \( f \theta \rightarrow_R t_1 \), then narrowing is guaranteed to find another substitution \( \sigma \) such that for some \( \rho \) we have \( \theta \rightarrow_{R} \rho \) and \( \sigma \) is more general than \( \rho \). In other words, \( t_1 \rightarrow_{R} t_2, t_3 \rightarrow_{R} t_5, \) and for some substitution \( \sigma, t_3 \theta \sigma =_{E} t_2 \).

3.2 Transformation rules

Let us define a FUN (Fold/Unfold Transforms) theory as a rewrite theory satisfying conditions (i) - (viii) together with (ix) sufficient completeness, which is required for the completeness of the unfold transformation, as shown below. A FUN transformation sequence of length \( k \) for a rewrite theory \((\Sigma, \Delta, \cup, B)\) is a sequence \((R_0, \ldots, R_i, R_{i+1}, \ldots, R_k)\), \( k \geq 0 \), where each \( R_j \) is a FUN theory, such that

- \( R_0 = (\Sigma, E_0, R_0), \) with \( E_0 = (\Delta \cup B) \) and \( R_0 = R \).

- For each \( 0 \leq j < i, R_{j+1} = (\Sigma, \Delta, \cup, B, R_j) \) is derived from \( R_j \) by an application of a transformation rule on the equation set \( \Delta_j \).

- For each \( 1 \leq j < k, R_{j+1} = (\Sigma, E_j, R_{j+1}) \) is derived from \( R_j \) by an application of a transformation rule on the rule set \( R_j \).

The transformation rules are introduction, elimination, fold, unfold, and abstraction, which are defined as follows.

**Definition Introduction.** We can obtain program \( R_{k+1} \) by adding to \( R_k \) a set of new equations (resp. rules), defining a new symbol \( f \) called eureka. We consider equations (resp. rules) of the form \( f(\overline{t}) = f'(\overline{t}) \rightarrow r \), such that:

- (1) \( f \) is a function symbol which does not occur in the sequence \( R_0, \ldots, R_k \) and is declared by \( f : s_1 \cdots s_n \rightarrow s [\Delta x] \), where \( s_1, \ldots, s_n, s \) are sorts declared in \( R_0 \) and \( \Delta x \) are equational attributes.

- (2) \( t_i \in T(\mathcal{C}, \mathcal{X}) \), and \( \varnothing \text{ar}(\overline{t}_i) = \varnothing \text{ar}(r_i) \), for all \( i \) - i.e., the equations/rules are non-erasing.

- (3) Every defined function symbol occurring in \( r_i \) belongs to \( R_0 \).

- (4) The set of new equations (resp. rules) are left linear, sufficient complete and non overlapping. For rules we require also right linearity.

\footnote{Properties (iv) and (v) can be achieved by a simple preprocessing of rewrite rules, while property (vi) is guaranteed by the disjointness of the sets of defined symbols, as shown in Appendix B.}

In general, the main idea consists of introducing new auxiliary function symbols which are defined by means of a set of equations/rules whose bodies contain a subset of the functions that appear in the right-hand side of an equation/rule that appears in \( R_0 \), whose definition is intended to be improved by subsequent transformation steps. The non overlapping property and the left linearity ensure confluence of eurekas, which is needed to preserve the completeness of the fold operation and will be discussed later. Sufficient completeness is needed to ensure the completeness of unfolding and will be discussed later. Right linearity on rules is needed to ensure narrowing completeness [30], and left linearity is also needed to preserve the right linearity of rules when doing folding. Consider, for instance, the folding of rule \( f(x) \rightarrow g(x) \) using the (non left linear) eureka \( new(x, x) \rightarrow g(x) \), which would produce a new rule \( f(x) \rightarrow new(x, x) \) which is not right linear.

Note that, once a transformation is applied to a eureka, the obtained equation/rule is not considered to be a eureka anymore. As we will see later, this is important for the folding operation, since we can only fold non-eureka equations/rules using eureka ones.

**Definition Elimination.** Let \( R_k \) be the rewrite theory \((\Sigma_k, \Delta_k, \cup, B_k, R_k)\). We can obtain program \( R_{k+1} \) by deleting from program \( R_k \),

- all equations that define the functions \( f_0, \ldots, f_n \), say \( \Delta' \), such that \( f_0, \ldots, f_n \) do not occur either in \( R_0 \) or in \( (\Sigma_k, (\Delta_k \cup \Delta') \cup B_k, R_k) \).

- all rules that define the functions \( f_0, \ldots, f_n \), say \( R' \), such that \( f_0, \ldots, f_n \) do not occur either in \( R_0 \) or in \( (\Sigma_k, \Delta_k \cup B_k, R_k \setminus R') \).

Note that the deletion of the equations/rules that define a function \( f \) implies that no function calls to \( f \) are allowed afterwards. However, subsequent transformation steps (in particular, folding steps) might introduce those deleted functions in the rhs’ of the equations/rules, thus producing inconsistencies in the resulting programs. To avoid this, we forbid any folding step after a definition elimination has been performed (this generally boils down to postpone all elimination steps to the end of the transformation sequence).

**Unfolding.** Let \( F \in R_k \) be an equation of the form \( l \rightarrow r \). We can obtain program \( R_{k+1} \) from program \( R_k \) by replacing \( F \) with the set of equations

\[ \{ l \sigma = r' \mid r \sim_{\sigma, R_k, B_k} r' \text{ \text{is a narrowing step in } E_k} \}. \]

Note that the narrowing steps are performed by using only the equations implicitly oriented as rewrite rules.

Let \( F \in R_k \) be a rule of the form \( l \rightarrow r \). We say that another rule \( R \rightarrow r' \rightarrow r' \in R_k \) is evil for \( F \), if there exist a substitution \( \rho \) and position \( p \in N \forall \mathcal{Pos}(l') \) such that \( r = l'_{p} \rho \). If in \( R_k \) there is no evil rule for \( F \), then we can obtain program \( R_{k+1} \) from program \( R_k \) by replacing \( F \) with the set of rules

\[ \{ l \sigma = r' \mid r \sim_{\sigma, R_k, \Delta_k, B_k} r' \text{\text{is a narrowing step in } R_k} \}. \]

The specified property of evil rules makes them unusable to unfold the selected rule because they do not allow a narrowing step from \( r \). The evil rule constraint is needed to guarantee the correctness of unfolding as shown in the following example.
EXAMPLE 3.1. Consider the following rewrite theory
\( \mathcal{R} = (\Sigma_{\mathcal{R}}, \emptyset, R) \), where \( \Sigma_{\mathcal{R}} \) is the signature containing all the symbols of \( R \) and

\[
R : \quad g_1(x) \rightarrow x \quad \quad \quad R' : \quad g_1(x) \rightarrow x
\]

\[
h(0) \rightarrow 1 \quad \quad \quad \quad h(0) \rightarrow 0
\]

\[
h(1) \rightarrow 0 \quad \quad \quad \quad h(1) \rightarrow 0
\]

\[
h(s(x)) \rightarrow 0 \quad \quad \quad \quad h(s(x)) \rightarrow 0
\]

\[
h(s_1(x)) \rightarrow 1 \quad \quad \quad \quad h(s_1(x)) \rightarrow 1
\]

\[
g_2(x) \rightarrow s_2(x) \quad \quad \quad \quad g_2(x) \rightarrow s_2(x)
\]

We get program \( R' = (\Sigma_{\mathcal{R}}, \emptyset, R') \) from \( R \) by applying an unfolding step over the rule defining function \( g_2 \) in \( R \), performing the narrowing step \( g_2(x) \leadsto s(x) \). Term \( h(g_2(0)) \) can be rewritten in \( R' \) to the normal forms 0 or 1 by means of the rewrite sequences \( h(g_2(0)) \rightarrow h(s_1(0)) \rightarrow h(s(0)) \rightarrow 0 \), and \( h(g_2(0)) \rightarrow h(s_1(0)) \rightarrow 1 \). The only possible rewrite sequence from \( h(g_2(0)) \) in \( R' \) is \( h(g_2(0)) \rightarrow h(s_1(0)) \rightarrow 0 \), missing solution 1. This is due to the rule \( h(s_2(x)) \rightarrow 1 \) which is evil for \( g_2(x) \rightarrow s_2(x) \) because the rhs \( s_2(x) \) is embedded in the lhs \( h(s_2(x)) \).

The use of narrowing empowers the unfold operation by implicitly embedding the instantiation rule (the operation of the Burstall and Darlington framework [9] that introduces an instance of an existing rule) into unfolding by means of unification.

It is worth noting that the rewrite theory has to be sufficiently complete in order to preserve the completeness of unfolding, as witnessed by the following example.

EXAMPLE 3.2. Consider the following rewrite theory
\( \mathcal{R} = (\Sigma_{\mathcal{R}}, \emptyset, R) \), where \( \Sigma_{\mathcal{R}} \) is the signature containing all the symbols of \( R \) and

\[
R = \left\{ \begin{array}{c}
f(0) \rightarrow 0 \\
g(x) \rightarrow s(f(x)) \\
h(x) \rightarrow 0 \\
h(s(x)) \rightarrow s(0) \\
\end{array} \right\} \quad \quad \quad R' = \left\{ \begin{array}{c}
f(0) \rightarrow 0 \\
g(0) \rightarrow s(0) \\
h(0) \rightarrow 0 \\
h(s(x)) \rightarrow s(0) \\
\end{array} \right\}
\]

We get program \( R' = (\Sigma_{\mathcal{R}}, \emptyset, R') \) from \( R \) by applying an unfolding step over the rule defining function \( g \) in \( R \), performing the narrowing step \( s(f(x)) \leadsto x=0 s(0) \). Now, the term \( h(g(s(0))) \) can be rewritten in \( R' \) to the normal form \( s(0) \), whereas this is no longer possible in the transformed program. This is due to the non-sufficient completeness of \( \mathcal{R} \), in particular to the partial definition of \( f \) so that the term \( f(s(0)) \) is not reducible in \( R \). Hence, by unfolding the call \( f(x) \), we improperly impose an unnecessary restriction in the domain of the function \( g \).

**Folding.** Let \( F \in \mathcal{R}_k \) be an equation (the "folded equation") of the form \( l = r \), and let \( F' \in \mathcal{R}_i, 0 \leq i \leq k \), be an equation (the "folding equation") of the form \( l' = r' \), such that \( r'|_p = b_{r'} r' \sigma \) for some position \( p \in \mathcal{N}_{\mathcal{V}P_{\mathcal{O}s}(r)} \) and substitution \( \sigma \). Note that, since we transform the equations of an equational theory, we consider here the congruence relation \( =_{\mathcal{B}_{\sigma}} \) modulo the equational axioms \( \mathcal{B}_{\sigma} \) (assuming an empty equation set). This is because we cannot consider a congruence modulo an equational theory which is being modified. Moreover, the following conditions must be satisfied:

1. \( F \) is not a eureka.
2. \( F' \) is a eureka.
3. The substitution \( \sigma \) is sort decreasing, i.e., if \( x \in \mathcal{X} \), then \( x\sigma \in \mathcal{T}_x(\mathcal{X}) \) such that \( s' \leq s \).
4. Let \( l' = f(t_m) \) and \( r'|_p = e \) and let \( f(t_m) \) and \( e \) have type \( s_f \) and \( s_e \), respectively; then \( s_f \leq s_e \).

Then, we can obtain program \( \mathcal{R}_{k+1} \) from program \( \mathcal{R}_k \) by replacing \( F \) with the new equation \( l = r'|_{\sigma|_p} \).

Folding can be applied to rules whenever the transformation of the equational theory has been completed. To fold rules we proceed as follows. Let \( F \in \mathcal{R}_k \) be a rule (the "folded rule") of the form \( (l \rightarrow r) \), and let \( F' \in \mathcal{R}_j \), \( 0 \leq j \leq k \), be a rule (the "folding rule") of the form \( (l' \rightarrow r') \), such that \( r'|_p = E_{r'} r' \sigma \) for some position \( p \in \mathcal{N}_{\mathcal{V}P_{\mathcal{O}s}(r)} \) and substitution \( \sigma \), fulfilling conditions (1) - (4) above. Then, we can obtain program \( \mathcal{R}_{k+1} \) from program \( \mathcal{R}_k \) by replacing \( F \) with the new rule \( (l \rightarrow r'|_{\sigma|_p}) \). Note that in this case we use the congruence modulo the equational theory \( E_{k} \) since it does not change any more after this stage.

The need for conditions (1) and (2) is twofold. These conditions forbid self-folding, that is, a folding operation with \( F = F' \), thus a rule with the same left and right-hand side cannot be produced, which may introduce infinite loops on derivations and destroy the correctness properties of the transformation system. These conditions also forbid the folding of a eureka, which is meaningless as illustrated in the following example.

EXAMPLE 3.3. Consider the following two rules:

\[
\begin{align*}
\text{new} & \rightarrow f \quad \text{(eureka)} \\
g & \rightarrow f \quad \text{(non-eureka)}
\end{align*}
\]

Without conditions (1) and (2), a folding of the eureka rule would be possible, obtaining the new rule \( \text{new} \rightarrow g \), which is nothing more than a redefinition of the symbol \( \text{new} \). Since transformation rules aim to optimize the original program with the support of eurekas, a folding over a eureka is meaningless or even dangerous.

Finally, conditions (3) and (4) ensure the sort compatibility of both the applied substitutions and the term that is inserted into the folded equation/rule right-hand side.

When presenting the definition introduction operation, we said that eurekas have to be confluent in order to ensure the completeness of the fold operation. We now discuss this point by means of an example.

EXAMPLE 3.4. Consider the following rewrite theory
\( \mathcal{R} = (\Sigma_{\mathcal{R}}, \emptyset, R) \), where \( \Sigma_{\mathcal{R}} \) is the signature containing all the symbols of \( R \) and

\[
R : \quad \begin{array}{c}
f(a,b) \rightarrow g(a,b) \\
(f(x,y)) \rightarrow (g(x,y)) \\
m(a) \rightarrow m(a) \\
m(b) \rightarrow m(b) \\
g(a,x) \rightarrow g(a,x) \\
g(b,x) \rightarrow g(b,x)
\end{array} \quad \quad \quad R' : \quad \begin{array}{c}
f(a,b) \rightarrow g(m(a),b) \\
f(a,b) \rightarrow g(m(a),b) \\
m(a) \rightarrow m(a) \\
m(b) \rightarrow m(b) \\
g(a,x) \rightarrow m(a) \\
g(b,x) \rightarrow m(b)
\end{array}
\]

We get program \( R' = (\Sigma_{\mathcal{R}}, \emptyset, R') \) from \( R \) by applying a folding step to the rule \( f(a,b) \rightarrow g(a,b) \) using the eureka \( m(a) \rightarrow a \). It is easy to see that in \( R' \) we can reduce term \( f(a,b) \) to the normal forms \( a \) or \( b \), while in \( R \) we can reach only the normal form \( a \). The point is that in \( R \), term \( f(a,b) \) can reduce only to \( g(a,b) \) while the fold operation introduces the possibility of rewriting to \( g(b,a) \) if the eureka defining \( m \) is not confluent. This leads to a new solution \( b \), thus missing the completeness.

**Abstraction.** The set of rules presented so far constitutes the core of our transformation system; however let us mention another useful rule, called abstraction, which can be simulated in our settings by applying appropriate definition introduction and folding steps. This rule is usually required to implement tupling, and it consists of replacing, by a new function, multiple occurrences of the same expression \( e \) in the right-hand side of an equation/rule. For instance,
consider the following equation
\[
double\_sum(x, y) = sum(sum(x, y), sum(x, y))
\]
where \(e = sum(x, y)\). The equation can be transformed into the following pair of equations
\[
double\_sum(x, y) = ds\_aux(sum(x, y))
\]
\[
ds\_aux(z) = sum(z, z)
\]
These equations are generated from the original one by a definition introduction of the eureka \(ds\_aux\) and then by folding the original equation by means of the newly generated eureka.

Note that the abstraction rule applies on equations or rules which are not right linear, since the same expression \(e\) occurs more than once in their rhs. Since we ask for rules to be right linear for introduction of the eureka

The following example demonstrates that the theorem above cannot consider the non-ground term \(g\). In \(R\), we have a (one step) derivation from term \(g(x)\) to the normal form \(s(f(x))\), whereas in \(R'\) there is no derivation starting from term \(g(x)\). So, the reduct \(s(f(x))\) is preserved by the transformation. The same example also shows that not even a more restricted non-ground semantics, such as the non-ground normal form semantics, is preserved. Nevertheless, in the reachability context of rewrite theories where confluence or termination are not required, this semantics is neither reasonable nor useful.

### 3.3 Correctness of the transformation system

Given a rewrite theory \(R\), let us define the set \(\text{red}_R = \{s \mid \exists t \in T_R \land t \rightarrow^*_R s\}\), as the set of all ground terms reachable in \(R\) in a finite number of rewrite steps (also called ground reducts). Let also be \(\text{GNF}_R = \{s \mid \exists t \in T_R \land t \rightarrow^*_R s \subseteq \text{red}_R\}\) the set of all ground normal forms reachable in \(R\).

Theorem 3.1 states the main theoretical result for the transformation system based on the elementary rules introduced so far: definition introduction, definition elimination, unfolding, folding, and abstraction. The result is strong correctness of a transformation sequence, i.e., the semantics of the ground reducts is preserved modulo the equational theory as stated by Theorem 3.1. The proof of the Theorem can be found in Appendix A.

**THEOREM 3.1.** Let \((R_0, \ldots, R_k), k > 0,\) be a FUN transformation sequence. Then, \(\text{GNF}_{E_0} =_B \text{GNF}_{E_k}, \text{GNF}_{R_0} =_E \text{GNF}_{R_k},\) and for all \(t \in T_{E_0},\) if \(s \in \text{red}_{R_0}(t)\) then there exist \(s_1, s_2\) such that \(s_1 \in \text{GNF}_{R_k}(t), s_2 \in \text{GNF}_{R_0}(s)\) and \(s_1 =_E s_2\). Viceversa, for all \(t \in T_{E_0},\) if \(s \in \text{red}_{R_0}(t)\) then there exist \(s_1, s_2\) such that \(s_1 \in \text{GNF}_{R_k}(t), s_2 \in \text{GNF}_{R_0}(s)\) and \(s_1 =_E s_2\).

The following example demonstrates that the theorem above cannot be lifted to the non-ground semantics of reducts.

**EXAMPLE 3.5.** Consider the FUN theory \(R = (\Sigma_R, \emptyset, R)\) where

\[
R = \begin{cases}
  f(0) & \rightarrow 0 \\
  f(s(x)) & \rightarrow s(x) \\
  g(x) & \rightarrow s(f(x))
\end{cases}
\]

\[
R' = \begin{cases}
  f(0) & \rightarrow 0 \\
  f(s(x)) & \rightarrow s(x) \\
  g(0) & \rightarrow s(0) \\
  g(s(x)) & \rightarrow s(s(x))
\end{cases}
\]

We get the rewrite theory \(R' = (\Sigma_R, \emptyset, R')\) from \(R\) by applying an unfolding step over the rule defining function \(g\) in \(R\). Then, consider the non-ground term \(g(x)\). In \(R\), we have a (one step) derivation from term \(g(x)\) to the normal form \(s(f(x))\), whereas in \(R'\) there is no derivation starting from term \(g(x)\). So, the reduct \(s(f(x))\) is preserved by the transformation. The same example also shows that not even a more restricted non-ground semantics, such as the non-ground normal form semantics, is preserved. Nevertheless, in the reachability context of rewrite theories where confluence or termination are not required, this semantics is neither reasonable nor useful.

### 4. CCT via Program Transformation

In this section, we explain how the transformation system presented so far can be employed to implement our CCT approach. The CCT methodology consists of several steps, which are illustrated in Figure 4, and summarized below.

1. **Defining Requirements** (Code Consumer). The code consumer provides the requirements to the code producer in the form of a rewrite theory, specifying the functions of interest with a naïve, non-optimized, even redundant piece of code. The rewrite theory can be written in Maude [14], a high-level specification language that implements rewriting logic [29].

2. **Defining New Functions** (Code Producer). The code producer has to generate an efficient implementation of the specified functions and a proof that such an implementation satisfies the required specifications. To this aim, the code producer uses the fold/unfold-based transformation system presented in Section 3 to (semi-)automatically obtain an efficient implementation of the specified functions. Moreover, some specific strategies such as composition and tupling can be easily automated (see [1] for more details). Subsequently, rather than sending the efficient functions as actual code to the consumer, the producer will send only a certificate consisting of a compact representation of the transformation rule sequence employed to derive the program. The strong correctness of the transformation system ensures that the obtained program is correct w.r.t. the initial consumer specifications, so the code producer does not need to provide extra proofs.

3. **Code Extraction** (Code Consumer). Assuming the transformation infrastructure is publicly available, once the certificate is received, the code consumer can apply the transformation sequence, described in the certificate, to the requirements, and the final program can be obtained without the need of other auxiliary software for the code extraction.

**Definition 4.2** (Certificate). Let \((R_0, \ldots, R_k), k > 0,\) be a transformation sequence. The certificate associated with the transformation sequence \((R_0, \ldots, R_k)\) is the ordered list of transfor-
mation rule descriptions \((d_1, \ldots, d_k)\) associated with the transformation rules \(r_1, \ldots, r_k\) s.t. \(\forall i \in \{1, \ldots, k\}, r_i\) is the transformation rule applied to \(R_{i-1}\) to obtain \(R_i\).

Let us show some selected examples to illustrate this.

**Example 4.1.** Let us now consider a simple specification of the Fibonacci function which uses the usual Peano notation to represent natural numbers. The specification is modeled by means of the following native equational theory.

\[
\begin{align*}
  \text{op} \ fib & : \text{Nat} \rightarrow \text{Nat} . \\
  (E1) \ & \ eq \ fib(0) = S(0) . \\
  (E2) \ & \ eq \ fib(S(0)) = S(0) . \\
  (E3) \ & \ eq \ fib(S(S(n))) = \text{fib}(S(n)) + \text{fib}(n) . 
\end{align*}
\]

Due to the highly recursive nature of this definition of \(\text{fib}\), the evaluation of \(\text{fib}(S^50(0))\) will compute many nested function calls again and again, and it will expand the original term into a whole binary tree of additions before collapsing it to a number. The exponential number of repeated function calls makes the evaluation of an expression like \(\text{fib}(S^50(0))\) in a naive way inefficient. Let us transform the previous Fibonacci definitions before collapsing it to a number. The exponential number of repeated function calls makes the evaluation of \(\text{fib}\) with the above rule very inefficient. Let us transform the previous Fibonacci definition into a more efficient one by using the tupling strategy.

1. First, we introduce the following eureka which makes use of the Pair data structure

\[
\begin{align*}
  \text{sort} \ Pair . \\
  \text{op} \ (.,.) : \text{Nat} \times \text{Nat} \rightarrow \text{Pair} . \\
  \text{op} \ aux : \text{Nat} \rightarrow \text{Pair} . \\
  (E4) \ & \ eq \ aux(0) = (\text{fib}(S(0)),\text{fib}(n)) .
\end{align*}
\]

(2) We now unfold the redex \(\text{fib}(S(n))\) of equation \((E4)\).

\[
\begin{align*}
  (E5) \ & \ eq \ aux(0) = (S(0),\text{fib}(0)) . \\
  (E6) \ & \ eq \ aux(S(n)) = (\text{fib}(S(n)) + \text{fib}(n),\text{fib}(S(n))) .
\end{align*}
\]

We unfold once again equation \((E5)\) in order to remove the call to \(\text{fib}\).

\[
(E7) \ eq \ aux(0) = (S(0),S(0)) .
\]

(3) Then, abstraction is applied to equations \((E3)\) and \((E6)\) by means of two new eurekas

\[
\begin{align*}
  \text{op} \ aux2 : \text{Pair} \rightarrow \text{Nat} . \\
  \text{op} \ aux3 : \text{Pair} \rightarrow \text{Pair} . \\
  (E8) \ & \ eq \ aux2(x,y) = x+y . \\
  (E9) \ & \ eq \ aux3(x,y) = (x+y,x) .
\end{align*}
\]

The second step for the abstraction is the folding of equations \((E3)\) and \((E6)\) by means of eurekas \((E8)\) and \((E9)\) respectively.

\[
\begin{align*}
  (E10) \ & \ eq \ \text{fib}(S(S(n))) = \text{aux2}(\text{fib}(S(n)),\text{fib}(n)) . \\
  (E11) \ & \ eq \ aux(S(n)) = aux3((\text{fib}(S(n)),\text{fib}(n))) .
\end{align*}
\]

(4) Finally, the right-hand sides of both equations are folded using the original definition of function \(aux\).

\[
\begin{align*}
  (E12) \ & \ eq \ \text{fib}(S(S(n))) = \text{aux2}(aux(n)) . \\
  (E13) \ & \ eq \ aux(S(n)) = aux3(aux(n)) .
\end{align*}
\]

The transformed (linear) definition of the equational theory for \(\text{fib}\) is as follows.

\[
\begin{align*}
  (E1) \ & \ eq \ fib(0) = S(0) . \\
  (E2) \ & \ eq \ fib(S(0)) = S(0) . \\
  (E12) \ & \ eq \ \text{fib}(S(S(n))) = \text{aux2}(aux(n)) . \\
  (E13) \ & \ eq \ aux(S(n)) = aux3(aux(n)) .
\end{align*}
\]

The resulting certificate \(C\) is as follows.

\[
C = (\text{Intro}(\text{op} aux : \text{Nat} \rightarrow \text{Pair} .), \quad \text{eq aux(n) = (fib(S(n)),fib(n)), Unfold(E4,[1]), Unfold(E5,[2]), Intro(op aux2 : \text{Pair} \rightarrow \text{Nat} ., \quad \text{eq aux2}(x,y) = x+y .), \quad \text{Intro(op aux3 : \text{Pair} \rightarrow \text{Pair} ., \quad \text{eq aux3}(x,y) = (x+y,x)), Fold(E8,E3,A), \quad \text{Fold}(E9,E6,A), Fold(E4,E10,[1]), Fold(E4,E11,[1])).}
\]

**Example 4.2.** Suppose the code consumer needs a function for computing the sum of the Fibonacci values of the natural numbers in a list. The type of the list of natural numbers is predefined in Maude. The consumer specification is a rewrite theory which consists of the equational theory of Example 4.1 defining the Fibonacci function along with the following set of rules.

\[
\begin{align*}
  \text{op} \ \text{sum-list} : \text{NatList} \rightarrow \text{Nat} . \\
  (R1) \ & \ \text{rl} \ \text{sum-list}(nil) \Rightarrow 0 . \\
  (R2) \ & \ \text{rl} \ \text{sum-list}(x \text{xs}) \Rightarrow \text{fib}(x) + \text{sum-list}(\text{xs}) .
\end{align*}
\]

The equational theory defining the Fibonacci function can be optimized as shown in Example 4.1. The above rules defining the
sum-list function can be transformed in a more efficient tail-recursive structure by using our fold/unfold framework as follows. (1) We first introduce the following definition

\[
\text{op sum-list-aux} : \text{NatList Nat} \rightarrow \text{Nat}.
\]

(3) Now, by folding rule (R2) and (R5) using the eureka (R3), we obtain the final tail-recursive program.

\[
\begin{align*}
\text{(R1)} & \quad \text{rl sum-list(nil)} \Rightarrow R1, \\
\text{(R6)} & \quad \text{rl sum-list}(x, \text{nil}) \Rightarrow R6, \\
\text{(R4)} & \quad \text{rl sum-list-aux}(nil, x) \Rightarrow \text{sum-list-aux}(x, \text{nil}), \\
\text{(R7)} & \quad \text{rl sum-list-aux}(x, \text{sum-list-aux}(x, \text{fib}(x) + y)) \Rightarrow \text{sum-list-aux}(x, \text{fib}(x) + y).
\end{align*}
\]

The certificate is then as follows.

\[
\text{C = (Intro}(\text{op sum-list-aux} : \text{NatList Nat} \rightarrow \text{Nat}), \\
\text{(rl sum-list-aux}(x, \text{fib}(x)) \Rightarrow \text{sum-list-aux}(x, \text{fib}(x) + y))
\]

By applying now the certificate to the initial specification, the code consumer can efficiently obtain the required efficient implementation.

5. Implementation

We implemented the transformation framework presented in Section 3 in a prototypical system, which consists of about 500 lines of code, written in Maude. Basically, our system allows us to perform the elementary transformation rules over a given initial program. The prototype also provides a simple user interface that shows a menu with the available operations as well as the program resulting after applying each transformation rule. Therefore, it is possible to see the complete transformation sequence. A snapshot of the interface is shown in Figure 5.

In order to implement the transformation rules, we made use of a useful Maude property called reflection. Rewriting logic is reflective in a precise mathematical way. In other words, there is a finitely presented rewrite theory \( U \) that is universal in the sense that we can represent in \( U \) any finitely presented rewrite theory \( R \) (including \( U \) itself) as a meta-term \( \bar{R} \), any term \( t \), \( t' \) in \( R \) as meta-terms \( \bar{t} \), \( \bar{t'} \), and any pair \( (R, t) \) as a meta-term \( \langle \bar{R}, \bar{t} \rangle \), in such a way that we have the following equivalence: \( t \rightarrow t' \) in \( R \) iff \( \langle \bar{R}, \bar{t} \rangle \rightarrow \langle \bar{R}, \bar{t'} \rangle \) in \( U \). Thanks to Maude reflection, our program transformation methodology has been easily implemented by manipulating the meta-term representations of rules and equations. In practice, transformation rules presented in Section 3 have been implemented as rewrite rules that work and manipulate the meta-term representation of the rewrite theories we want to transform. On the other hand, by virtue of our reflective design, our rewrite theory for program transformation is also available to the level of the CCT infrastructure, which allows us to reuse it in a clear and principled way.

Since the unfolding operation uses narrowing, we employed the META-E-NARROWING module, which is part of the Full Maude distribution [39]. The narrowing implemented in Maude is called narrowing with simplification because it combines the narrowing relation with rewriting to normal form (represented by \( \rightarrow^* \)). The combined relation \( \rightarrow^* \) is defined as \( \rightarrow^* \), the combined relation \( \rightarrow^* \) is defined as the transitive closure of \( \rightarrow^* \).

6. Conclusions

In this paper, we provide a novel rewriting logic framework for Code Carrying Theory, which is fed with a novel narrowing-based, fold/unfold program transformation methodology. The core transformation rules are folding, unfolding, definition introduction, definition elimination, and abstraction. The correctness of the program transformation framework guarantees that the transformed program is equivalent to the initial one in the sense of Theorem 3.1 (Section 3.3). As already mentioned, this framework opens up new applications to program optimization, program synthesis, program specialization, and theorem proving for first-order typed rule-based languages such as ASF+SDF, Elan, OBJ, CafeOBJ, and Maude.

We have shown that our methodology can be effectively applied to CCT and may significantly simplify the code producer task. Actually, by applying a strategy such as composition, tupling, etc., the code producer can (semi-)automatically obtain the final improved program. Of course, for the methodology to pay off in practice, the transformation system could be instrumented to also provide an estimation of the achieved optimization. However, this is out of the scope of this paper. We intend to investigate such an extension in a future work. Thanks to the fact that our transformation methodology relies on narrowing, this can be done by adapting the narrowing-based transformation strategies of [1]. Moreover, as an outcome of the transformation process, a compact representation of the sequence of applied transformation rules is delivered as a certificate to the code consumer. The code consumer needs only applying the certificate to the initial requirements to obtain the desired program. This checking can be completely automated.

We provided a prototypical implementation of the transformation framework written in Maude. Some available Maude formal tools can be employed to verify relevant program properties required by our methodology such as termination [19] of the equation set, confluence [20] and sufficient completeness [24] of rewrite theories. We are now implementing the complete CCT infrastructure along with several fully automatic transformation strategies on top of the transformation system. Moreover, we are developing a number of transformation templates in the sense of [10] for rewriting logic theories, whose correctness should (hopefully) follow from our results easily, while [10] uses an automated theorem prover for the verification.

References


Figure 2. Snapshot of the transformation system interface written in Maude.
A.1. Let $E, E'$ be two equational theories satisfying properties (i) - (iv), (viii) and (ix), such that $E'$ is obtained from $E$ by a fold operation over an equation $E$. Then $\text{GF}_{E'} \equiv_{B} \text{GF}_{E}$.

PROOF A.2. Let $E$ be an equation of the form $(l = r)$ while the eureka $E_e$, used to fold equation $E$, be of the form $(l = r_j)$. From the definition of the fold operation it follows that $t_{p} \equiv_{p} r_{p} \sigma$ for some position $p \in N \cup \text{Pos}(r)$ and the result of the folding operation of $E$ by $E_e$ is the equation $E_f$: $(l = r[t_{\sigma}]_{p})$. Finally, $\Delta' = \Delta - \{E\} \cup \{E_f\}$.

\begin{itemize}
  \item[\textbf{A.1.}] We want to prove that, given any ground term $t$, if $t \rightarrow_{\Delta, B} s$ then $t \rightarrow_{\Delta', B} s'$ and $s \equiv_{B} s'$. We will prove it by induction on the length of the rewriting sequence in $\Delta$.
    \begin{itemize}
      \item[(n = 0)] This case is immediate since $t \equiv_{B} s$.
      \item[(n > 0)] Let us decompose the rewriting sequence form $t$ to $s$ as follows: $t \rightarrow t_1 \rightarrow \ldots \rightarrow t_n \equiv_{B} s$. On the rewriting sequence from $t_1$ to $s$ we can apply the induction hypothesis and we now concentrate on the first rewriting step. If $t$ rewrites to $t_1$ without using equation $E$, the same step can be performed in $\Delta'$ and the claim holds. Otherwise, if equation $E$ is used for the first step, it means that (i) $t_{p} \equiv_{B} s_{1}$ and (ii) $t_1 = t[r_{\sigma}]_{p}$. Then, considering term $t$ and equation $E_f$, we note that, by (i), it is possible in $E'$ a rewriting step from $t$ using $E_f$, thus obtaining term $t_2 = t[r_{\sigma}]_{p} \sigma$. Propagating substitution $\sigma$, we obtain $t_3 = t[r_{\sigma}]_{p} \sigma$. Since term $t_2$ is embedded in $t_3$, a rewriting step using $E_e$ is possible and we obtain $t_3 = t[r_{\sigma}]_{p} \sigma \equiv_{B} s$. We have that $t_3 \equiv_{B} t[r_{\sigma}]_{p} \sigma \equiv_{B} t[r_{\sigma}]_{p} \equiv_{B} t_1$, which completes the proof.
    \end{itemize}
\end{itemize}

\begin{itemize}
  \item[\textbf{A.2.}] We want to prove that, given any ground term $t$, if $t \rightarrow_{\Delta, B} s$ then $t \rightarrow_{\Delta', B} s'$ and $s \equiv_{B} s'$. We will prove it by induction on the length of the rewriting sequence in $\Delta'$.
    \begin{itemize}
      \item[(n = 0)] This case is immediate since $t \equiv_{B} s'$.
      \item[(n > 0)] Let us decompose the rewriting sequence form $t$ to $s'$ as follows: $t \rightarrow t_1 \rightarrow \ldots \rightarrow t_n \equiv_{B} s'$. If $t$ rewrites to $t_1$ without using equation $E_f$, the same step can be performed in $\Delta$ and, by applying the induction hypothesis on the rewriting sequence from $t_1$ to $s'$, the claim holds. Otherwise, if equation $E_f$ is used for the first step, note that term $t_1$ will embed an instance of the left-hand side ($l_c$) of equation $E_f$. If the following rewrite step from $t_1$ in $\Delta'$ uses the equation $E_e$, we can show how the considered rewrite step $t \rightarrow_{E_f} t_1 \rightarrow_{E_e} t_2$ can be simulated in $\Delta$ by only one rewrite step from $t$ using equation $E$. While, if the rewrite step form $t_1$ does not use equation $E_e$, since $E'$ is confluent, we can consider a different rewrite sequence from $t$ to $s'$ where $E_e$ is used to rewrite $t_1$, and then use the induction on the length of this new rewrite sequence.
    \end{itemize}
\end{itemize}

\begin{itemize}
  \item[\textbf{A.3.}] Let $E, E'$ be two equational theories satisfying properties (i) - (iv), (viii) and (ix), such that $E'$ is obtained from $E$ by a unfold operation over an equation $E$. Then $\text{GF}_{E'} \equiv_{B} \text{GF}_{E}$.

PROOF A.3. Let $E$ be an equation of the form $(l = r)$, while $E_e$ is the set of equations used to unfold $E$, each one of the form $(l = r_j)$ for $i = 0, \ldots, k$. Let $f_1, \ldots, f_n$ with $n \leq k$ be the set of symbols defined by equations $E_1, \ldots, E_k$. Also let $r \rightarrow_{\Delta, B} r'_j$ (where $j \in \{1, \ldots, n\}$) be the $\Delta, B$-narrowing step such that the
result of unfolding \( E \) using \( E \) is the equation \( E_{\theta}^{*} : (l\sigma) = r_{\theta}^{*} \).
From the definition and the correctness of narrowing, we recall that:

1. \( \forall j. r\sigma_{j} \rightarrow_{E} r_{\theta}^{*} \)
2. \( \forall j \) there exists position \( p_{j} \in NVP(\theta) \) such that \( r_{\theta}^{*}[p_{j}] = B \)
3. \( \forall j. r_{\theta}^{*} = (r[p_{j}])\sigma_{j} \)

\( \square \) We want to prove that, given any ground term \( t \), if \( t \rightarrow^{\Delta}_{\theta} \) \( s \) then \( t \rightarrow^{\Delta} B \) \( s' \), and \( s = \sigma s' \). From \( t \rightarrow^{\Delta} B \) \( s' \) and the confluence and termination of \( E \), there exists a rewrite sequence from \( t \) to \( s \) where at each step the left-most inner-most redex is reduced. We will prove the result by induction on the length of such a rewrite sequence.

\( (n = 0.) \) This case is immediate since \( \sigma s = s' \).

\( (n > 0.) \) Let us decompose the rewriting sequence form \( t \) to \( s \) as follows: \( t \rightarrow t_{1} \rightarrow \cdots \rightarrow t_{n} \). On the rewrite sequence from \( t_{1} \) to \( s \) we can apply the induction hypothesis and we now concentrate on the first rewriting step. If \( t \) rewrite to \( t_{1} \) without using \( R \), the same step can be performed in \( \Delta' \) and the claim holds. Otherwise, if \( R \) is used for the first step, it means that (i) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), (ii) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), (iii) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), and (iv) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \). Note that from (ii) and (iv) it follows that if \( t \) rewrite to \( t_{1} \) without using rule \( R \), the same step can be performed in \( \Delta' \) and the claim holds. Otherwise, if rule \( R \) is used for the first step, it means that (i) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), (ii) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \). Then, considering term \( t \) and rule \( R\), we note that, by (i), it is possible in \( \Delta' \) a rewriting step from \( t \) using \( R \), thus obtaining a term \( t_{1} = E t \sigma \theta \). Propagating substitution \( \sigma \) we obtain \( t_{1} = E t \sigma \theta \). Since term \( l \) is embedded in \( t_{1} \), a rewriting step using \( R \) is also possible and we obtain \( t_{1} = E t \sigma \theta \). We have that \( t_{1} = E t \sigma \theta \), which completes the proof.

\( \square \) We want to prove that, given any ground term \( t \), if \( t \rightarrow^{\Delta}_{\theta} s' \) then \( t \rightarrow^{\Delta} \sigma s' \), and \( s = \sigma s' \). We will prove it by induction on the length of the rewriting sequence in \( \Delta' \).

\( (n = 0.) \) This case is immediate since \( \sigma s = s' \).

\( (n > 0.) \) Let us decompose the rewriting sequence form \( t \) to \( s \) as follows: \( t \rightarrow t_{1} \rightarrow \cdots \rightarrow t_{n} \). On the rewriting sequence from \( t_{1} \) to \( s \) we can apply the induction hypothesis and we now concentrate on the first rewriting step. If \( t \) rewrite to \( t_{1} \) without using rule \( R \), the same step can be performed in \( \Delta' \) and the claim holds. Otherwise, if rule \( R \) is used for the first step, it means that (i) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), (ii) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \). Then, considering term \( t \) and rule \( R \), we note that, by (i), it is possible in \( \Delta' \) a rewriting step from \( t \) using \( R \), thus obtaining a term \( t_{1} = E t \sigma \theta \). Propagating substitution \( \sigma \) we obtain \( t_{1} = E t \sigma \theta \). Since term \( l \) is embedded in \( t_{1} \), a rewriting step using \( R \) is also possible and we obtain \( t_{1} = E t \sigma \theta \). We have that \( t_{1} = E t \sigma \theta \), which completes the proof.

\( \square \) We want to prove that, given any ground term \( t \), if \( t \rightarrow^{\Delta}_{\theta} \) \( s' \) then \( t \rightarrow^{\Delta} \sigma s' \), and \( s = \sigma s' \). We will prove it by induction on the length of the rewriting sequence in \( \Delta' \).

\( (n = 0.) \) This case is immediate since \( \sigma s = s' \).

\( (n > 0.) \) Let us decompose the rewriting sequence form \( t \) to \( s \) as follows: \( t \rightarrow t_{1} \rightarrow \cdots \rightarrow t_{n} \). On the rewriting sequence from \( t_{1} \) to \( s \) we can apply the induction hypothesis and we now concentrate on the first rewriting step. If \( t \) rewrite to \( t_{1} \) without using rule \( R \), the same step can be performed in \( \Delta' \) and the claim holds. Otherwise, if rule \( R \) is used for the first step, it means that (i) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), (ii) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \). Then, considering term \( t \) and rule \( R \), we note that, by (i), it is possible in \( \Delta' \) a rewriting step from \( t \) using \( R \), thus obtaining a term \( t_{1} = E t \sigma \theta \). Propagating substitution \( \sigma \) we obtain \( t_{1} = E t \sigma \theta \). Since term \( l \) is embedded in \( t_{1} \), a rewriting step using \( R \) is also possible and we obtain \( t_{1} = E t \sigma \theta \). We have that \( t_{1} = E t \sigma \theta \), which completes the proof.

**Proof**

Let \( R \) be a rule of the form \( (l \rightarrow \) while the eureka \( R_{e} \), used to fold rule \( R \), be of the form \( (l_{e} \rightarrow r_{e}) \). From the definition of the fold operation it follows that \( r_{\theta}^{*} = E R_{e} \theta \) for some position \( p \in NVP(\theta) \) and the results of the folding operation of \( R \) by \( R_{e} \) is the rule \( R_{f} : (l \rightarrow \) let us decompose the rewriting sequence form \( t \) to \( s \) as follows: \( t \rightarrow t_{1} \rightarrow \cdots \rightarrow t_{n} \) on the rewriting sequence from \( t_{1} \) to \( s \) we can apply the induction hypothesis and we now concentrate on the first rewriting step. If \( t \) rewrite to \( t_{1} \) without using rule \( R \), the same step can be performed in \( \Delta' \) and the claim holds. Otherwise, if rule \( R \) is used for the first step, it means that (i) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \), (ii) \( t \rightarrow_{\Delta} B \) \( \sigma \theta \). Then, considering term \( t \) and rule \( R \), we note that, by (i), it is possible in \( \Delta' \) a rewriting step from \( t \) using \( R \), thus obtaining a term \( t_{1} = E t \sigma \theta \). Propagating substitution \( \sigma \) we obtain \( t_{1} = E t \sigma \theta \). Since term \( l \) is embedded in \( t_{1} \), a rewriting step using \( R \) is also possible and we obtain \( t_{1} = E t \sigma \theta \). We have that \( t_{1} = E t \sigma \theta \), which completes the proof.

**Lemma A.3.** Let \( R, R' \) be two FUN theories such that \( R' \) is obtained from \( R \) by a fold operation over a rule \( B \). Then \( GNF_{R} = E GNF_{R'} \).

<table>
<thead>
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<th>Destination</th>
<th>Description</th>
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<td>11</td>
<td>2010/11</td>
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</table>
LEMMA A.4. Let \( R, R' \) be two FUN theories such that \( R' \) is obtained from \( R \) by an unfold operation over a rule \( R \). Then \( \text{GNF}_R = \# \text{GNF}_R' \).

PROOF A.5. Let \( R^k \) be a rule of the form \((l \rightarrow r)\), and \( R_1, \ldots, R_k \) the sets of rules used to unfold rule \( R^k \), each one of the form \((l_i \rightarrow r_i)\) for \( i = 1, \ldots, k \). Let \( f_1, \ldots, f_m \) with \( n \leq k \) the set of symbols defined by rules \( R_1, \ldots, R_k \). Also let \( r \sim_{\sigma_1, R, \Delta, B} r' \), \( j \in \{1, \ldots, n\} \), be the \( R \cup \Delta, B \)-narrowing step such that the result of unfolding \( R^k \) using \( R_j \) is the rule \( R_j^k = \{(\sigma_j \rightarrow r_j')\} \). From the definition and the correctness of narrowing, we recall that:

1. \( \forall j \cdot r \sigma_j \rightarrow R_j^k \)
2. \( \forall j \) there exists position \( p_j \in \Delta \text{VPos}(r) \) such that \( r[p_j] \sigma_j = B \)
3. \( \forall j \cdot r'[p_j] \sigma_j \)

We want to prove that, given any ground term \( t \), if \( t \rightarrow R^k \) then \( t \rightarrow R^k \), and \( s \rightarrow R^k \). We will prove it by induction on the length of rewriting sequence in \( R \).

(n = 0.) This case is immediate since \( t = e \).

(n > 0.) Let us decompose the rewrite sequence form \( t \rightarrow s \) as follows: \( t \rightarrow R_1 \rightarrow \cdots \rightarrow R_k \rightarrow s \). On the rewrite sequence from \( t \) to \( s \) we can apply the induction hypothesis and we now concentrate on the first rewrite step. If \( t \) writes to \( t_1 \) without using rule \( R^k \), the same step can be performed in \( R' \) and the claim holds. Otherwise, we want to describe a procedure to reorder an initial fragment of the rewrite sequence from \( t \) to \( s \) in such a way that it is then trivial to simulate it in \( R' \) and then use the induction hypothesis on the rest of the sequence. Consider a ground term \( w \) and a subsequent application of rules \( R^k \) and \( R_1 \) in \( R \) as follows. If \( w[p] = B \), then \( \{R^k\} \cup E \) we obtain an E-equivalent term to \( w[p\theta] \), which embeds a ground instance of \( r \). So, such term contains some occurrences of the symbols \( f_1, \ldots, f_n \). Then, if we can apply \( \{R_1\} \cup E \) for some \( j \in \{1, \ldots, n\} \) to reduce the redex having one such symbol as its root, we obtain an E-equivalent term to \( w[r[p\theta]] \theta \). The key point that we need is to note that such a subsequent application of rules \( R^k \) and \( R_1 \) in \( R \), can be simulated in \( R' \) by an application of rule \( R^k \). In fact, since the rewrite step using \( R_1 \) occurs at position \( p_j \in \Delta \text{VPos}(r) \), it follows that the left hand side \( \sigma_j \) of rule \( R_1 \) unifies with the subterm \( r[p_j] \) by substitution \( \sigma_j \) that subsumes \( \theta \), and therefore the narrowing step \( r \sim_{\sigma_j, R, \Delta, B} r'[p_j] \) can be proven in \( R \cup E \). By the definition of unfolding, the rule \( \sigma_j \rightarrow r_j' \) is one \( R^k \) belonging to \( R' \). Finally, by applying \( \{(R_j^k) \cup \Delta\} \) to term \( w \) we obtain an E-equivalent term to \( w[r[p\theta]] \theta \). The basic aim of the sequence reordering procedure \text{rorderSeq} of Figure A is to change the rule application order thus obtaining an equivalent sequence (in the sense that the same normal form \( s \) is reached) where the application of rule \( R^k \) is immediately followed by an application of a rule \( R_j \). In the procedure a rewrite sequence is represented as a list of rewrite steps \((R, p)\) where \( R \) is the applied rule and \( p \) the position of the reduced redex. Each rewrite step is intended to be followed by a \( \Delta, B \) normalization. The procedure takes as input the rewrite sequence starting from the rewrite step using rule \( R^k \) and returns the reordered rewrite sequence. List \( s_1 \) contains the reordered portion of the sequence which can be easily simulated in \( R' \) while \( s_2 \) contains the rest of the sequence (if any). The auxiliary procedure \text{rorder} uses two auxiliary lists \( s_1 \) and \( s_2 \). The former contains the sequence of steps that are moved before \((R^k, p)\), while the latter contains the skipped steps during the reordering.
reorderSeq((R^n, p) : seq) = let (s1, s2) = reduce((R^n, p), [ ], [ ], seq) in merge(s1, s2)
reorder((R^n, p), ns, vs, (R, q) : ts)
if q = p_j ∈ ∑ Pos(r) and R = R_j, then return ([(ns, (R^n, p), (R_j, p_j)), [vs, ts]])
if q ∉ ∑ Pos(r) > p and R = R^n, then let (ns1, ts1) = reorder((R^n, q), [ ], [ ], ts) in reorder((R^n, p), [ns, ns1], vs, ts1)
if q ∉ ∑ Pos(r) > p, then reorder((R^n, p), [ns, (R, q)], vs, ts)
if q ≠ p and q ≠ p_j, then reorder((R^n, p), ns, [vs, (R, q)], ts)
if q < p and f_1, ..., f_n do not appear in the resulting term, then return ([(R, q)], [vs, ts])
if q < p and R = R^n, then let (ns1, ts1) = reorder((R^n, p), ns, vs, ts) in reorder((R^n, q), [ns, ns1], vs, ts1)
if q < p, then reorder((R^n, p), [ns, (R, n)], vs, ts)

Figure 3. Rewrite sequence reordering procedure.

sequence from t_1 to s. Hence we can use the inductive hypothesis on s_2.

**Exhaustiveness of the cases.** Note that the only case in which the considered rule R applies at a position q ≥ p and q ∈ ∑ Pos(r) can be the one where q = p_j for some j ∈ {1, ..., n} (case (1)). This is in fact the only possibility due to the evil rule constraint considered in the unfolding operation. In other words, if R applies at a position q ∈ ∑ Pos(r) and q ≠ p_j for all j, then R is a evil rule for R^n and that means that the unfolding operation would not have been performed.

We want to prove that, given any ground term t, if t →^1_{R^r} s′ then t →_{R^s} s, and s = b.s′. We can prove by induction on the length of the rewriting sequence in \( R^r \).

LEMMA A.5. Let (\( R_0, ..., R_k, k > 0 \)) be a virtual FUN transformation sequence. Then GNF_{E_0} = B GNF_{E_1}, and GNF_{R_0} = E_0 GNF_{R_1}.

**Proof.** The proof follows immediately from Lemma A.1 and A.2, and Lemma A.3 and A.4, since, at each of the first i-th transformation steps (0 ≤ i ≤ k) a fold or unfold operation has been performed over the equations, and at each of the following k - i transformation steps, a fold or unfold operation has been performed over rules.

**Lemma A.6.** For all \( R, R' \) be two FUN theories such that \( R' \) is obtained from \( R \) by a fold or unfold operation over a rule R. Then, for all \( t \in T_0 \), if \( s \in red_{R}(t) \) then there exist \( s_1, s_2 \) such that \( s_1 \in GNF_{R}(t), s_2 \in GNF_{R'}(s) \) and \( s = E, s_2 \). Vice versa, for all \( t \in T_0 \), if \( s \in red_{R'}(t) \) then there exist \( s_1, s_2 \) such that \( s_1 \in GNF_{R'}(t), s_2 \in GNF_{R}(s) \) and \( s = E, s_2 \).

**Proof.** We will prove by induction on the length of the rewrite sequence from \( t \) to \( s \) in \( R \).

Finally, since each FUN transformation sequence can be transformed into an equivalent virtual FUN transformation sequence (following Definition A.1) which produces the same output program, we have the following theorem by combining Theorem A.2 and Lemma A.6.

**Corollary A.1 (Strong Correctness).** Let (\( R_0, ..., R_k, k > 0 \)) be a FUN transformation sequence. Then, GNF_{E_0} = B GNF_{E_1}, GNF_{R_0} = E_0 GNF_{R_1}, and for all \( t \in T_0 \), if \( s \in red_{R}(t) \) then there exist \( s_1, s_2 \) such that \( s_1 \in GNF_{R}(t), s_2 \in GNF_{R_0}(s) \) and \( s = E, s_2 \). Vice versa, for all \( t \in T_0 \), if \( s \in red_{R'}(t) \) then there exist \( s_1, s_2 \) such that \( s_1 \in GNF_{R'}(t), s_2 \in GNF_{R}(s) \) and \( s = E, s_2 \).

**B. Coherence and Consistence**

We propose here a method to guarantee the coherence of \( \rightarrow_{A,B} \) with B and the E-consistency of \( \rightarrow_{A,B} \) with B when associativity (A), commutativity (C) and unity (U) axioms (or a subset of them) are considered. The procedure consists of adding some extension
variables to each equation or rule having in the left-hand side a topmost symbol which is declared with a subset of the \{A, C, U\} axioms, thus obtaining a new set of generalized rules.

For instance, assume we declare the operator + and provide it with only the associativity axiom. Consider now a rule \( r : x + x \to 0 \), and a term \( t : a + (a + b) \). Then \( \to_{r,B} \) is not \( A \)-coherent since there is no matching between term \( t \) and the left-hand side of \( r \), whereas the \( A \)-equivalent term \( t' : (a + a) + b \) matches the left-hand side by means of substitution \( \{x/a\} \). To make \( \to_{r,B} \) \( A \)-coherent we need to add some extension variables \( y \) and \( z \) thus producing the following set of rules:

\[
\begin{align*}
x + x &\to 0 \\
x + x + y &\to 0 + y \\
z + x + x &\to z + 0 \\
z + x + x + y &\to z + 0 + y 
\end{align*}
\]

Now, given any term \( t \) with topmost symbol +, \( t \) admits a rewriting step with one of these rules iff for each term \( t' \) which is \( A \)-equivalent to \( t \), \( t' \) admits a rewriting step too. If operator + is declared with both \( A \) and \( C \) axioms, then to make \( \to_{r,B} \) \( AC \)-coherent we need to introduce only the following two rules:

\[
\begin{align*}
x + x &\to 0 \\
x + x + y &\to 0 + y 
\end{align*}
\]

Finally, if also the identity axiom (U) is declared for +, then only rule

\[
x + x + y \to 0 + y
\]

is needed for \( ACU \)-coherence. Now it is quite easy to derive the needed generalization for any subset of axioms \{A, C, U\}. A similar transformation can be defined for the case of the equations.

In Maude, this generalization does not have to be performed explicitly as a transformation of the specification, because it is achieved implicitly in a built-in, automated way.

For what concern the \( \to_{R,B} E \)-consistence with \( \to_{\Delta,B} \) we want to show how this is guaranteed by the disjointness of the sets of defined symbols \( D_1 \) and \( D_2 \) (see Sec 2) and the fact that symbols in \( D_1 \) can not appear in the lhs of rewrite rules. Consider a rewrite theory and (i) a rewrite step \( t_1 \to_{R,B} t_2 \) using a rule \( R : l \to r \) and (ii) \( t_1 \to_{\Delta,B} t_3 \). From (i) it follows that there exist substitution \( \theta \) and position \( p \) such that \( t_1|_p =_{B} l\theta \) and \( t_2 =_{\Delta,B} t_1[r\theta]|_p \). Since symbols in \( D_1 \) can not appear in \( l \), all possible steps using \( \Delta, B \) from \( t_1 \) can not modify the structure of \( l \) embedded in \( t_1 \), that will then appear unmodified in \( t_3 \). So, after the \( \Delta, B \) steps, there would exists substitution \( \rho =_{\Delta,B} \theta \) such that \( t_3|_p =_{B} l\rho \) and hence it is possible a rewrite step \( t_3 \to_{R,B} t_4 \) such that \( t_4 =_{E} t_2 \).