



Note

Bilateral locally testable languages

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Abstract

We give an algebraic characterization of a new variety of languages that will be called *bilateral locally testable languages* and denoted as BLT. Given $k > 0$, the membership of a word x to a *BLT* (k -BT) language can be decided by means of exploring the segments of length k of x , as well as considering the order of appearance of those segments when we scan the prefixes and the suffixes of x . In this paper, we also characterize the syntactic semigroup of *BLT* languages in terms of the equations of the variety they belong to, as well as in terms of the join of two previously studied varieties.

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1. Introduction

The classification of Finite Idempotent Monoids (Bands) has been done in [6,19], while the classification of the families of languages associated to them has been studied in [15]. The classification of the families of languages corresponding to locally idempotent semigroups has not been completely established. Besides the locally testable (LT) languages, only the families of right and left locally testable languages have been studied [7].

The family of *LT* languages is, together with the families of star-free [13] and piecewise testable languages [16], one of the most studied in the literature about formal languages. *LT* languages were first studied by McNaughton [10] and algebraically characterized as the variety of languages which correspond to \mathbf{LJ}_1 , the variety of locally

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idempotent and locally commutative finite semigroups by Brzozowski and Simon [4] and by Zalcstein [20].

Given an integer $k > 0$, we say that L is k -testable (k - T) if given a word $x \in L$, any other word having the same prefix and suffix of length $k - 1$ and containing exactly the same segments of length k than x also belongs to L . A language is LT if it is k -testable for some value of k .

Several variations of the family of LT languages have appeared in the literature. One of them consists in dropping the condition for prefixes and suffixes in the definition. Those languages are called *strongly locally testable* and have been characterized by Beauquier and Pin in [3], while the Strongly Locally Testable semigroups (concepts that do not correspond each other), have been characterized by Selmi in [14]. The fact of counting the number of occurrences of the segments of length k in the words up to a certain threshold has been studied in [17,18].

The order of appearance of segments when we explore the words from left to right (resp. from right to left) has been recently considered [7]. These families of languages are, respectively, called, *right locally testable* (RLT) and *left locally testable* (LLT) and the corresponding varieties of semigroups are the varieties of locally idempotent and locally R -trivial (resp. L -trivial) finite semigroups, which will be denoted as \mathbf{LR}_1 and \mathbf{LL}_1 . It has been shown [7] that \mathbf{LJ}_1 is the intersection \mathbf{LR}_1 and \mathbf{LL}_1 . They also have the “strongly” version [12]. Another family of languages which extends the family of LT are the languages described below.

Informally, for a given value of $k > 0$ a language L is called *k -Bilateral Testable* (k - BT) if for a word $x \in L$, any other word y belongs to L if it satisfies the following condition:

- The order of appearance of the first occurrences of the segments of length k when we explore the string y from left to right and also from right to left is the same as when we explore x .

A language L is called *Bilateral Locally Testable* (BLT) if it is k - BT for some value of k .

This new family of languages strictly includes the families of right and left locally testable languages [7] which also includes the family of locally testable languages.

The algebraic characterization of this family of languages is solved by proving a theorem that shows that one of the conditions of a theorem on graphs (Simon) [5] can be relaxed if we consider the order of appearance of the edges in the paths. We prove that the variety of finite semigroups corresponding to the variety BLT is $\mathbf{L}(\mathbf{R}_1 \vee \mathbf{L}_1)$, that is, the localization of $\mathbf{R}_1 \vee \mathbf{L}_1$ [2], also known as \mathbf{MNB} [1], this latter variety being defined by the equations $x^2 = x$ and $xyxzx = x.yzx$.

As all varieties of bands are local [9], it is easily seen that $\mathbf{L}(\mathbf{R}_1 \vee \mathbf{L}_1)$ coincides with the join $\mathbf{LR}_1 \vee \mathbf{LL}_1$.

2. Preliminaries and notation

We refer the reader to [1,5,8,11] for terms not defined here.

2.1. Automata, languages and semigroups

Let Σ be a finite alphabet and let Σ^* be the free monoid generated by Σ with concatenation as the internal law and λ as neutral element. A *language* L over Σ is a subset of Σ^* . The length of a word x is denoted by $|x|$, while Σ^k represents the set of all words of length k over Σ . Given $k > 0$, the prefix (resp. suffix) of length $k - 1$ of a word $x \in \Sigma^{k-1}\Sigma^*$ is denoted $i_k(x)$ (resp. $f_k(x)$) whereas the set of segments of length k of x is denoted $t_k(x)$. $\text{Pr}(x)$ (resp. $\text{Suf}(x)$) denotes the set of prefixes (resp. suffixes) of x .

A deterministic finite automaton (DFA) is a quintuple $A = (Q, \Sigma, \cdot, q_0, F)$ where Q is a finite set of states, Σ is a finite alphabet, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states and \cdot is a partial function that maps $Q \times \Sigma$ in Q , which can be easily extended to words. A word x is accepted by an automaton A if $q_0 \cdot x \in F$. The set of words accepted by A is denoted by $L(A)$.

Given an automaton A , $\forall a \in \Sigma$, we can define the function $a^A : Q \rightarrow Q$ as $qa^A = q \cdot a$, $\forall q \in Q$. For $x \in \Sigma^*$, the function $x^A : Q \rightarrow Q$ is defined inductively: λ^A is the identity on Q and $(xa)^A = x^A a^A$, $\forall a \in \Sigma$. Clearly, $\forall x, y \in \Sigma^*$, $(xy)^A = (x^A)^A (y^A)^A$. The set $\{a^A : a \in \Sigma\}$ is denoted by M_A . The set of functions $\{x^A : x \in \Sigma^+\}$ is a finite semigroup under the operation of composition of functions, and is denoted as S_A and called *semigroup of A*.

Let $L \subseteq \Sigma^*$ and let \equiv be an equivalence relation defined in Σ^* . The equivalence class of x in \equiv will be denoted as $[x]_{\equiv}$. We say that \equiv saturates L , if L is the union of equivalence classes modulo \equiv . An equivalence relation is called a congruence if it is both-sides compatible with the operation of the monoid.

A *labelled directed graph* G with labels in Σ is given by two sets, a finite set of *vertices* V and a finite set of *edges* $E \subset V \times \Sigma \times V$. The edge (p, a, q) will sometimes be denoted as $p \xrightarrow{a} q$. Two edges (p, a, q) and (r, b, s) are *consecutive* if $q = r$. The set of paths of G is the subset of words in E^+ that does not contain any segments of length two whose edges are non-consecutive. If C is the set of non-consecutive edges in G , the set of paths in G is $P = E^+ - E^* C E^*$. The function $\tau : P \rightarrow 2^E$ is defined by the following conditions:

$$\begin{aligned} \tau(e) &= \{e\} \quad \text{if } e \in E, \\ \tau(xe) &= \tau(x) \cup \tau(e) \quad \forall x \in P, \forall e \in E. \end{aligned}$$

For each path x , $\tau(x)$ gives the set of edges traversed by x without regard to order or multiplicity.

2.2. Varieties of finite semigroups and languages

For every $L \subseteq \Sigma^*$, the congruence \sim_L defined as $x \sim_L y \Leftrightarrow (\forall u, v \in \Sigma^*, uxv \in L \Leftrightarrow uyv \in L)$, is called the *syntactic congruence* of L and it is the coarsest congruence that saturates L . Σ^* / \sim_L is called the *syntactic monoid of L* and is denoted as $S(L)$. The morphism $\varphi : \Sigma^* \rightarrow S(L)$ that maps each word to its equivalence class modulo \sim_L is called the *syntactic morphism* of L .

Given a semigroup S , the set of idempotents of S is denoted $E(S)$. If $e \in E(S)$, eSe is called the *local submonoid* associated to e .

A *variety of finite monoids* is a class of finite monoids closed under morphic images, submonoids and finite direct products.

A *variety of recognizable languages* is a class of languages closed under finite union and intersection complement, inverse morphisms, and, right and left quotients. Eilenberg [5] proved that varieties of finite monoids and varieties of languages are in one-to-one correspondence. If \mathbf{V} is a variety of semigroups, we denote as \mathbf{LV} the variety of semigroups whose local submonoids belong to \mathbf{V} and $\mathcal{L}_V(\Sigma^*)$ is the variety of languages over Σ whose syntactic semigroup lies in \mathbf{V} .

If M and N are monoids, $M * N$ denotes the semidirect product of M and N [5]. Given varieties of finite monoids \mathbf{V} and \mathbf{W} , we denote by $\mathbf{V} * \mathbf{W}$ the variety generated by all the semidirect products of the form $M * N$ with $M \in \mathbf{V}$, $N \in \mathbf{W}$. The semidirect product of varieties of finite monoids is associative.

3. BLT languages

We recall [7] $\equiv_{k,R}$ (resp. $\equiv_{k,L}$) denotes the congruence that defines the family of right (resp. Left) locally testable languages. For every $x, y \in \Sigma^*$, if $|x| < k$, $x \equiv_{k,R} y$ if and only if $x = y$, otherwise $x \equiv_{k,R} y$ if and only if $f_k(x) = f_k(y)$ and for every $u \in Pr(x)$ (resp. $Pr(y)$) there exists $v \in Pr(y)$ (resp. $Pr(x)$) such that $t_k(u) = t_k(v)$.

The relation $\equiv_{k,L}$ is defined in a similar way, by replacing the prefixes with the suffixes of x and y in the above definition. A language L is k -RT (resp. k -LT) if it is saturated by the relation $\equiv_{k,R}$ (resp. $\equiv_{k,L}$). L is RLT (resp. LLT) if it is k -RT (resp. k -LT) for some value of $k \geq 1$. It has been shown [7] that the families of RLT and LLT languages are two varieties of languages whose intersection is the variety of locally testable languages.

We are going to define the congruence \equiv_k that will help us to define the family of BLT languages.

3.1. The congruence \equiv_k

Definition 1. For every $x, y \in \Sigma^*$ we define $x \equiv_k y$ if and only if $x \equiv_{k,R} y$ and $x \equiv_{k,L} y$.

Informally two words $x, y \in \Sigma^k$ are \equiv_k -equivalent if the order of appearance of new segments of length k in both words when they are explored from left to right and when they are explored from right to left is the same.

It is easily seen that the congruence \equiv_k is of finite index and refines the congruences $\equiv_{k,R}$ and $\equiv_{k,L}$.

Definition 2. We say that a language L is k -Bilateral Testable (k -BT) if it is saturated by the congruence \equiv_k . L is bilateral locally testable (BLT) if it is k -BT for some value of $k \geq 1$.

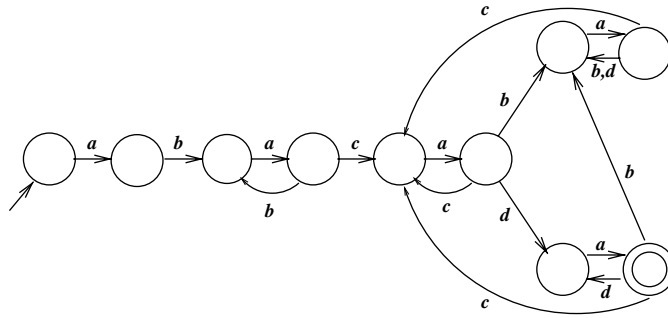


Fig. 1.

Example 3. The language recognized by the automaton A , $L(A)$, in Fig. 1 is $[abacada]_{\equiv_{1,R}} \cap [abacada]_{\equiv_{2,L}}$. It can be seen that $L(A)$ is BLT. The local submonoid associated to the idempotent \mathbf{ad} is $\{0, abad, ad, adacad\}$. Taking $x = abad$ and $y = adacad$ one can see that $xyx \neq xy$ so $L(A)$ is not right locally testable. The local submonoid associated to the idempotent \mathbf{ab} is $\{0, ab, abacab, abad\}$. Taking $x = abacab$ and $y = abad$ one can see that $xyx \neq yx$ so $L(A)$ is not left locally testable. Notice that the fact that $L(A)$ is not right locally testable can also be seen if you consider that for every value of $k \geq 1$, the word $(ab)^k(ac)^k ab(ad)^k(ac)^k(ad)^k a$ belongs to $L(A)$, while $(ab)^k(ac)^k ab(ad)^k(ac)^k(ad)^k(ac)^k ab(ad)^k a$ does not, and both words are $\equiv_{k,R}$ -equivalents. The fact that $L(A)$ is not left locally testable can be proved in a similar way.

Proposition 4. Let \equiv_k be the congruence over Σ^* defined above. Then

- (1) $\forall x \in \Sigma^{k-1}, \forall y, z \in \Sigma^* (xy = zx \Rightarrow xy \equiv_k xy^2)$.
- (2) $\forall x \in \Sigma^{k-1}, \forall y \in \Sigma^*$ we have $xyx \equiv_k xyxyx$.
- (3) $\forall x \in \Sigma^{k-1}, \forall y, z, w \in \Sigma^* xyxzx yxwx yx \equiv_k xyxzxwx yx$.

Proof. Item (1) follows from the fact that $xy \equiv_{k,R} xy^2$ and $xy \equiv_{k,L} xy^2$ (see [7]).

Item (2) follows by replacing in (1) the word y for yx , and z for xy .

In order to prove (3), we have that $xyxzx yx \equiv_{k,R} xyxzx$ and, as $\equiv_{k,R}$ is a congruence, we have $xyxzx yxwx yx \equiv_{k,R} xyxzxwx yx$. We similarly have that $xyxzx yxwx yx \equiv_{k,L} xyxzxwx yx$, so it follows that $xyxzx yxwx yx \equiv_k xyxzxwx yx$. \square

4. A theorem on graphs

Let \sim_{right} (resp. \sim_{left}) be the smallest graph congruence with the property that for any vertex p and for any cycles $p \xrightarrow{x} p$ around p . Then $x \sim_{\text{right}} x^2$ and $xyx \sim_{\text{right}} xy$ (resp. $x \sim_{\text{right}} x^2$ and $xyx \sim_{\text{left}} yx$).

Let \sim be the smallest graph congruence with the property that for any vertex p and for any cycles x, y, z around p . Then $x \sim x^2$ and $xyxzx \sim xyzx$.

Let \cong (resp. \cong_R, \cong_L) be the graph congruence defined as $x \cong y$ if $\tau(x) = \tau(y)$ and the order of appearance of the edges in x and in y when we explore the paths in both directions (resp. to the right, resp. to the left) is the same.

Then we have the following.

Theorem 5. $x \sim y$ if and only if $x \cong y$.

Proof. The equivalence between \sim_{right} (resp. \sim_{left}) and \cong_R (resp. \cong_L) has been proved in [7].

Both congruences \sim_{right} and \sim_{left} fulfil the equations $x = x^2$ and $xyxzx = xyzx$. As \sim is the smallest congruence that satisfies those equations, we have $x \sim y \Rightarrow x \sim_{\text{right}} y \wedge x \sim_{\text{left}} y \Rightarrow x \cong_R y \wedge x \cong_L y \Rightarrow x \cong y$.

For the converse, let us first suppose that x and y are cycles. We are going to prove that $x \sim_{\text{right}} y \Rightarrow x \sim yx$ by induction in the number of times that either $x^2 \sim_{\text{right}} x$ or $xyx \sim_{\text{right}} xy$ are applied. If y follows from x using a number m of times one of the above rules, we will use $\overset{m}{\sim}_{\text{right}}$ rather than \sim_{right} .

For $m = 1$ we have the following possibilities:

- (1) $x = x_1x_2x_3, y = x_1x_2x_2x_3$. Then we have that $yx = x_1x_2x_2x_3x_1x_2x_3 \sim x_1x_2x_3x_1x_2x_3 \sim x_1x_2x_3 = x$.
- (2) $x = x_1x_2x_2x_3, y = x_1x_2x_3$. Then we have that $yx = x_1x_2x_3x_1x_2x_2x_3 \sim x_1x_2x_3x_1x_2x_3 \sim x_1x_2x_3 = x$.
- (3) $x = x_1x_2x_3x_2x_4, y = x_1x_2x_3x_4$. Then we have $yx = x_1x_2x_3x_4x_1x_2x_3x_2x_4 \sim x_1x_2x_3x_2x_4x_1x_2x_3x_2x_4 \sim x_1x_2x_3x_2x_4 = x$.
- (4) $x = x_1x_2x_3x_4, y = x_1x_2x_3x_2x_4$. Then we have $yx = x_1x_2x_3x_2x_4x_1x_2x_3x_4 \sim x_1x_2x_3x_4x_1x_2x_3x_4 \sim x_1x_2x_3x_4 = x$.

Let us suppose that $x \overset{\leq n}{\sim}_{\text{right}} y \Rightarrow x \sim yx$.

If $x \overset{n+1}{\sim}_{\text{right}} y$ then $x \overset{n}{\sim}_{\text{right}} z \overset{1}{\sim}_{\text{right}} y$. By induction hypothesis $x \sim zx$ and $z \sim yz$. Then $x \sim zx \sim yzx$ and hence $x \sim yx$.

In an analogous way, we have $x \sim_{\text{left}} y \Rightarrow x \sim xy$. Finally, $x \sim_{\text{right}} y \wedge x \sim_{\text{left}} y \Rightarrow x \sim yx \wedge y \sim yx \Rightarrow x \sim y$.

If x and y are not cycles, we have $p \overset{x}{\underset{y}{\rightrightarrows}} q$ in a graph G . We add an edge $q \overset{a}{\rightarrow} p$ with the condition $q \overset{a}{\rightarrow} p \notin G, a \notin \Sigma$ and thus, we obtain a graph G' . In G' we have that $x \sim_{\text{right}} y \wedge x \sim_{\text{left}} y \Rightarrow xa \sim_{\text{right}} ya \wedge xa \sim_{\text{left}} ya$ and then $xa \sim ya$.

We only need to show that $xa \sim ya \Rightarrow x \sim y$. Again, by induction on the number of times that either $x^2 \sim x$ or $xyxzx \sim xyzx$ are applied, if $xa \overset{1}{\sim} ya$, as a only appears once, neither of the equations can be applied, so $x \sim y$. If $xa \overset{n+1}{\sim} ya$, then $xa \overset{n}{\sim} za \overset{1}{\sim} ya$. Using induction hypothesis $x \sim z$, and using the base $z \sim y, x \sim y$ holds. \square

5. Characterization of BLT languages

Definition 6. A semigroup S is central repetition free (crf) if $\forall x, y, z \in S, xyxzx = xyzx$.

Proposition 7. For every $k > 0$, Σ^+ / \equiv_k is locally idempotent and locally crf.

Proof. Let π_k be the projection from Σ^* to Σ^+ / \equiv_k . Let $e \in E(\Sigma^+ / \equiv_k)$ and let us consider $r, s, t \in \Sigma^+ / \equiv_k$. There exists $w \in \Sigma^k \Sigma^*$ such that $\pi_k(w) = e$ and $x, y, z \in \Sigma^*$ such that $\pi_k(x) = r, \pi_k(y) = s$ and $\pi_k(z) = t$.

- $ere = \pi_k(wxw) = \pi_k(wxwxw) = erere = (ere)(ere)$, so Σ^+ / \equiv_k is locally idempotent.
- $(ere)(ese)(ere)(ete)(ere) = eresereterere = \pi_k(wxwywxwzwxw) = \pi_k(wxwywxwzwxw) = ereseterere = (ere)(ese)(ete)(ere)$, so Σ^+ / \equiv_k is locally crf.

Theorem 8. A recognizable language $L \subset \Sigma^*$ is BLT if and only if $S(L)$ is locally idempotent and locally crf.

Proof. If L is BLT, then there exists $k \geq 1$ such that L is k -BT. Then Σ^+ / \equiv_k recognizes L and hence $S(L)$ divides Σ^+ / \equiv_k . As Σ^+ / \equiv_k is locally idempotent and locally crf, so is $S(L)$.

For the converse, let $M(L)$ be the syntactic monoid of L with $n = \text{Card}(M(L))$ and let $\varphi: \Sigma^* \rightarrow M(L)$ be the syntactic morphism of L . We are going to prove that L is $(n + 1)$ -BT.

Let $A = (Q, \Sigma, \cdot, q_0)$ be a semiautomaton, with $Q = \Sigma^n$, and where the transformations are defined as $a^A(a_1 a_2 \dots a_n) = a_2 \dots a_n a, \forall a_1 a_2 \dots a_n \in Q, \forall a \in \Sigma$. The semiautomaton A is known as the graph of n -factors.

If y, z, w are loops around p in A , let us see that

- (a) $py \sim_L py^2$
- (b) $pyzywy \sim_L pyzwy$.

Let $p = a_1 a_2 \dots a_n$. As $\text{Card}(M(L)) = n$, the elements $1, \varphi(a_1), \varphi(a_1 a_2), \dots, \varphi(a_1 a_2 \dots a_n)$ cannot all be different, so there exists $i < j$ such that $\varphi(a_1 \dots a_i) = \varphi(a_1 \dots a_j)$ and thus we have a factorization

$$p = rst \tag{1}$$

with $s \neq 1$ such that

$$rs \sim_L r. \tag{2}$$

We have two possibilities:

- $\varphi(s) = 1$. Then $S(L) = M(L)$ and $S(L)$ is idempotent and crf.
- $\varphi(s) \neq 1$. Then $\exists k \geq 1$ such that $x = s^k$ is idempotent. As $S(L)$ is locally idempotent and locally crf, it is aperiodic and then $\varphi(x) \neq 1$.

Let y, w, z be loops around p . As the words py, pw and pz end with p we can write

$$sty = y't, \quad stw = w't \text{ and } stz = z't \tag{3}$$

for segments y', w' and z' such that ry', rw' and rz' end with rs , and then write

$$ry' = y''rs, \quad rw' = w''rs \text{ and } rz' = z''rs. \tag{4}$$

As $rs \sim_L r$ we have that

$$ry's \sim_L ry', \quad rw's \sim_L rw' \text{ and } rz's \sim_L rz' \tag{5}$$

and thus

$$ry's^k = ry'x \sim_L ry'. \tag{6}$$

Proof of (a). $py^2 \stackrel{(1)}{=} rstyy \stackrel{(3)}{=} ry'ty \stackrel{(5)}{\sim_L} ry'sty \stackrel{(3)}{\sim_L} ry'y't \stackrel{(4)}{=} y''rsy't \stackrel{(2)}{\sim_L} y''ry't \stackrel{(6)}{=} y''ry'xt \stackrel{(2)}{\sim_L} y''rsy'xt \stackrel{(4)}{=} ry'y'xt.$

Then $py^2 \sim_L ry'y'xt \stackrel{(6)}{\sim_L} ry'xy'xt \stackrel{(2)}{\sim_L} rxy'xy'xt.$

$S(L)$ locally idempotent implies $rxy'xy'xt \sim_L rxy'xt \stackrel{(2)}{\sim_L} ry'xt \stackrel{(6)}{\sim_L} ry't \stackrel{(2)}{\sim_L} rsty \stackrel{(1)}{=} py.$ So $py \sim_L py^2.$

Proof of (b). $pyzywy \stackrel{(1)}{=} rstzywy \stackrel{(3)}{=} ry'tzywy \stackrel{(4)}{=} y''rstzywy \stackrel{(3)}{=} y''rz'tywy.$

Following the same sequence of computations with z, y and w we obtain

$$pyzywy = y''z''y''w''rsty \stackrel{(3)}{=} y''z''y''w''ry't \stackrel{(6)}{\sim_L} y''z''y''w''ry'xt \stackrel{(2)}{\sim_L} y''z''y''w''rsy'xt \stackrel{(4)}{=} y''z''y''rw'y'xt \stackrel{(6)}{\sim_L} y''z''y''rw'xy'xt.$$

Applying these computations repeatedly we obtain $pyzywy \sim_L rxy'xz'xy'xw'xy'xt.$

As $S(L)$ is locally crf we obtain $rxy'xz'xy'xw'xy'xt \sim_L rxy'xz'xw'xy'xt.$

$$\text{Using (2) and (4) repeatedly we obtain } rxy'xz'xw'xy'xt \stackrel{(2)}{\sim_L} ry'xz'xw'xy'xt \stackrel{(4)}{=} y''rsxz'xw'xy'xt \stackrel{(2)}{\sim_L} y''rz'xw'xy'xt \stackrel{(4)}{=} \dots y''z''w''y''rst \stackrel{(4)}{=} y''z''w''ry't \stackrel{(2)}{\sim_L} y''z''rw'y't.$$

$$\text{Finally, } y''z''rw'y't \stackrel{(3)}{=} y''z''rw'sty \stackrel{(5)}{\sim_L} y''z''rw'ty \stackrel{(3)}{=} y''z''rstwy \stackrel{(4)}{=} y''rz'twy \stackrel{(3)}{=} y''rstzw \stackrel{(4)}{=} ry'tzwy \stackrel{(3)}{=} rstzywy \stackrel{(1)}{=} pyzwy.$$

It follows that $pyzywy \sim_L pyzwy.$

Then, given $p \xrightarrow[z]{y} q$ with $x \cong y$ we have that $py \sim_L pz.$

Let $u, v \in \Sigma^*$ such that $u \equiv_{n+1} v$ and let $p = i_{n+1}(u), q = f_{n+1}(u).$ Let $u = py, v = pz.$ Then there exist two paths $p \xrightarrow[z]{y} q$ in which all the edges are of the form $r \xrightarrow{a} t,$ with $r, t \in t_{n+1}(u) = t_{n+1}(v).$ Besides, all the transitions appear in u and v in the same order from left to right and from right to left. Then $u \sim_L v,$ that is, L is BLT. \square

Finally, as all varieties of bands are local, it follows that $\mathbf{LV} = \mathbf{V} * \mathbf{LI}$ for any of $\mathbf{V} = \mathbf{R}_1, \mathbf{V} = \mathbf{L}_1$ or $\mathbf{V} = \mathbf{R}_1 \vee \mathbf{L}_1.$ As the semidirect product is left distributive with respect to the join operation [1], we have that $\mathbf{L}(\mathbf{R}_1 \vee \mathbf{L}_1) = (\mathbf{R}_1 \vee \mathbf{L}_1) * \mathbf{LI} = (\mathbf{R}_1 * \mathbf{LI}) \vee (\mathbf{L}_1 * \mathbf{LI}) = \mathbf{LR}_1 \vee \mathbf{LL}_1.$

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