Towards a dependency pair framework for context-sensitive rewriting

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Abstract
Termination is one of the most interesting problems when dealing with context-sensitive rewrite systems. Although there is a good number of techniques for proving termination of context-sensitive rewriting (CSR), the dependency pair approach, one of the most powerful technique for proving termination of rewriting, has been not investigated in connection with proofs of termination of CSR. In this paper, we investigate how to use dependency pairs in proofs of termination of CSR.

Key words: Dependency pairs, term rewriting, program analysis, termination.

1 Introduction

A replacement map is a mapping \( \mu : \mathcal{F} \to \mathcal{P}(\mathbb{N}) \) satisfying \( \mu(f) \subseteq \{1, \ldots, k\} \), for each \( k \)-ary symbol \( f \) of a signature \( \mathcal{F} \) [Luc98]. We use them to discriminate the argument positions on which the rewriting steps are allowed; rewriting at the topmost position is always possible. In this way, for a given Term Rewriting System (TRS), we obtain a restriction of rewriting which we call context-sensitive rewriting (CSR [Luc98,Luc02]). With CSR we can achieve a terminating behavior with non-terminating TRSs, by pruning (all) infinite rewrite sequences. Proving termination of CSR is an interesting problem with several applications in the fields of term rewriting and programming languages (see [DLMMU06,GM04,Luc02,Luc06] for further motivation).

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Several methods have been developed for proving termination of CSR under a replacement map $\mu$ for a given TRS $\mathcal{R}$ (i.e., for proving the $\mu$-termination of $\mathcal{R}$). In particular, a number of transformations which permit to treat termination of CSR as a standard termination problem have been described (see [GM04,Luc06] for recent surveys). Also, direct techniques like polynomial orderings and the context-sensitive version of the recursive path ordering have been investigated [BLR02,GL02,Luc05]. Up to now, however, the dependency pairs method [AG00,HM04], one of the most powerful tools for proving termination of rewriting, has been not investigated in connection with proofs of termination of CSR. In this paper, we address this problem.

Roughly speaking, given a TRS $\mathcal{R}$, the dependency pairs associated to $\mathcal{R}$ conform a new TRS $\mathcal{DP}(\mathcal{R})$ which (together with $\mathcal{R}$) determines the so called dependency chains whose finiteness or infiniteness characterize termination or non-termination of $\mathcal{R}$. The dependency pairs can be presented as a dependency graph, where the absence of infinite chains can be analyzed by considering the cycles in the graph. In this paper, we show that (with some restrictions) these basic intuitions are also valid for CSR.

**Example 1.1** Consider the following non-terminating TRS $\mathcal{R}$ borrowing the well-known Toyama’s example [GM04, Example 15]:

\[
\begin{align*}
f(a,b,x) & \rightarrow f(x,x,x) \\
c & \rightarrow a \\
c & \rightarrow b
\end{align*}
\]

Together with $\mu(f) = \{3\}$. As shown by Giesl and Middeldorp, among all existing transformations for proving termination of CSR, only the complete Giesl and Middeldorp’s transformation [GM04] (yielding a TRS $\mathcal{R}_{\mu}^{C}$) could be used in this case, but no concrete proof of termination for $\mathcal{R}_{\mu}^{C}$ is known yet. Furthermore, $\mathcal{R}_{\mu}^{C}$ has 13 dependency pairs and the dependency graph contains many cycles. In contrast, the CS-TRS has only one context-sensitive dependency pair and the corresponding dependency graph has no cycle!

**Example 1.2** Consider the following non-terminating TRS $\mathcal{R}$ [Luc02]:

\[
\begin{align*}
sqr(0) & \rightarrow 0 & add(0,x) & \rightarrow x \\
sqr(s(x)) & \rightarrow s(add(sqr(x),dbl(x))) & add(s(x),y) & \rightarrow s(add(x,y)) \\
first(0,x) & \rightarrow nil & half(0) & \rightarrow 0 \\
first(s(x),cons(y,z)) & \rightarrow cons(y,first(x,z)) & half(dbl(x)) & \rightarrow x \\
dbl(s(x)) & \rightarrow s(dbl(x)) & half(s(0)) & \rightarrow 0 \\
dbl(0) & \rightarrow 0 & half(s(s(x))) & \rightarrow s(half(x)) \\
terms(N) & \rightarrow cons(recip(sqr(N)),terms(s(N)))
\end{align*}
\]

which can be approximated to the value of $\pi^2/6$. In particular, when $\mu(cons) = \{1\}$ and $\mu(f) = \{1, \ldots, ar(f)\}$ for all other symbols $f$. $\mathcal{R}$ is $\mu$-terminating. Hirokawa and Middeldorp report on a proof of termination of $\mathcal{R}^{\mu}_{GM}$ (which corresponds to the incomplete Giesl and Middeldorp’s transformation for proving termination of CSR, see [GM04]) by using the dependency pairs approach [HM05, Appendix A]. In particular, termination of $\mathcal{R}^{\mu}_{GM}$ im-
plies the $\mu$-termination of $R$ above. As remarked by Hirokawa and Middeldorp, there are 33 dependency pairs in $R_{GM}^\mu$. However, the context-sensitive dependency graph of $R$ consists of seven pairs with only four cycles, each of them containing a single dependency pair. Moreover, as explained below each of them can be removed by using the adaptation to CSR of Hirokawa and Middeldorp’s subterm criterion [HM04] which we develop below. Thus, the system is $\mu$-terminating. This is in contrast with Hirokawa and Middeldorp’s proof which also use the subterm criterion (for $DP(R_{GM}^\mu)$) but without being able to remove all cycles, thus requiring to do some constraint solving.

After some preliminary definitions in Section 2, Section 3 introduces the general framework to compute and use context-sensitive dependency pairs for proving termination of CSR. Section 4 shows how to compute the (estimated) context-sensitive dependency graph. Section 5 adapts Hirokawa and Middeldorp’s subterm criterion to CSR. Section 6 concludes.

2 Preliminaries

Throughout the paper, $X$ denotes a countable set of variables and $F$ denotes a signature, i.e., a set of function symbols $\{f, g, \ldots\}$, each having a fixed arity given by a mapping $ar : F \to \mathbb{N}$. The set of terms built from $F$ and $X$ is $T(F, X)$. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of $t$. Given positions $p, q$, we denote its concatenation as $p.q$. If $p$ is a position, and $Q$ is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. We denote the empty chain by $\Lambda$. The set of positions of a term $t$ is $Pos(t)$. The subterm at position $p$ of $t$ is denoted as $t|_p$ and $[s]_p$ is the term $t$ with the subterm at position $p$ replaced by $s$. We write $t \triangleright t$ if $s = t|_p$ for some $p \in Pos(t)$ and $t \triangleright s$ if $t \triangleright s$ and $t \neq s$. The symbol labelling the root of $t$ is denoted as root$(t)$. A context is a term $C \in T(F \cup \{\Box\}, X)$ with zero or more ‘holes’ $\Box$ (a fresh constant symbol).

A rewrite rule is an ordered pair $(l, r)$, written $l \rightarrow r$, with $l, r \in T(F, X)$, $l \notin X$ and $Var(r) \subseteq Var(l)$. The left-hand side (lhs) of the rule is $l$ and $r$ is the right-hand side (rhs). $A$ TRS is a pair $R = (F, R)$ where $R$ is a set of rewrite rules. Given $R = (F, R)$, we consider $F$ as the disjoint union $F = C \cup D$ of symbols $c \in C$, called constructors and symbols $f \in D$, called defined functions, where $D = \{\text{root}(l) \mid l \rightarrow r \in R\}$ and $C = F - D$.

Context-sensitive rewriting.

A mapping $\mu : F \to P(\mathbb{N})$ is a replacement map (or $F$-map) if $\forall f \in F, \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [Luc98]. Let $M_F$ be the set of all $F$-maps (or $M_R$ for the $F$-maps of a TRS $(F, R)$). A binary relation $R$ on terms is $\mu$-monotonic if $t R s$ implies $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) R f(t_1, \ldots, t_{i-1}, s, \ldots, t_k)$ for every $t, s, t_1, \ldots, t_k \in T(F, X)$. The set of $\mu$-replacing positions $\text{Pos}^\mu(t)$ of $t \in T(F, X)$ is: $\text{Pos}^\mu(t) = \{\Lambda\}$, if $t \in X$ and $\text{Pos}^\mu(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(\text{root}(t))} i \cdot \text{Pos}^\mu(t|i)$,
if \( t \not\in X \). The set of replacing variables of \( t \) is \( \mathcal{V} \text{ar}^\mu(t) = \{ x \in \mathcal{V} \text{ar}(t) \mid \exists p \in \mathcal{P} \text{os}^\mu(t), tl_p = x \} \). The \( \mu \)-replacing subterm relation \( \supseteq^\mu \) is given by \( t \supseteq^\mu s \) if there is \( p \in \mathcal{P} \text{os}^\mu(t) \) such that \( s = tl_p \). We write \( t \supseteq^\mu s \) if \( t \supseteq^\mu s \) and \( t \neq s \). In context-sensitive rewriting (CSR [Luc98]), we (only) contract replacing redexes: \( t \) \( \mu \)-rewrites to \( s \), written \( t \hookrightarrow^\mu s \) (or \( t \hookrightarrow^\mu s \)), if \( t \supseteq^\mu s \) and \( p \in \mathcal{P} \text{os}^\mu(t) \). A TRS \( \mathcal{R} \) is \( \mu \)-terminating if \( t \hookrightarrow^\mu s \) is terminating. A term \( t \) is \( \mu \)-terminating if there is no infinite \( \mu \)-rewrite sequence \( t = t_1 \hookrightarrow^\mu t_2 \hookrightarrow^\mu \cdots \hookrightarrow^\mu t_n \hookrightarrow^\mu \cdots \) starting from \( t \). A pair \( (\mathcal{R}, \mu) \) where \( \mathcal{R} \) is a TRS and \( \mu \in M_\mathcal{R} \) is often called a CS-TRS.

Dependency pairs.

Given a TRS \( \mathcal{R} = (\mathcal{F}, R) = (C \uplus D, R) \) a new TRS \( \mathcal{D} \mathcal{P}(\mathcal{R}) = (\mathcal{F}^\sharp, D(R)) \) of dependency pairs for \( \mathcal{R} \) is given as follows: if \( f(t_1, \ldots, t_m) \rightarrow r \in R \) and \( r = C[g(s_1, \ldots, s_n)] \) for some defined symbol \( g \in D \) and \( s_1, \ldots, s_n \in \mathcal{T} \mathcal{F}(\mathcal{F}, X) \), then \( f^\sharp(t_1, \ldots, t_m) \rightarrow g^\sharp(s_1, \ldots, s_n) \in D(R) \), where \( f^\sharp \) and \( g^\sharp \) are new fresh symbols (called tuple symbols) associated to defined symbols \( f \) and \( g \) respectively [AG00]. Let \( D^\sharp \) be the set of tuple symbols associated to symbols in \( D \) and \( \mathcal{F}^\sharp = \mathcal{F} \uplus D^\sharp \). As usual, for \( t = f(t_1, \ldots, t_k) \in \mathcal{T} \mathcal{F}(\mathcal{F}, X) \), we write \( t^\sharp \) to denote the marked term \( f^\sharp(t_1, \ldots, t_k) \). Given \( T \subseteq \mathcal{T} \mathcal{F}(\mathcal{F}, X) \), let \( T^\sharp \) be the set \( \{ t^\sharp \mid t \in T \} \).

A reduction pair \((\succeq, \sqsubseteq)\) consists of a stable and weakly monotonic quasi-ordering \( \succeq \), and a stable and well-founded ordering \( \sqsubseteq \) satisfying either \( \succeq \circ \sqsubseteq \subseteq \sqsubseteq \) or \( \sqsubset \circ \succeq \subseteq \sqsubseteq \). Note that monotonicity is not required for \( \sqsubseteq \).

3 Context-Sensitive Dependency Pairs

Let \( M_{\infty, \mu} \) be a set of minimal non-\( \mu \)-terminating terms in the following sense: \( t \) belongs to \( M_{\infty, \mu} \) if \( t \) is non-\( \mu \)-terminating and every strict \( \mu \)-replacing subterm \( s \) (i.e., \( t \supseteq^\mu s \)) is \( \mu \)-terminating. Obviously, if \( t \in M_{\infty, \mu} \), then \( \text{root}(t) \) is a defined symbol. A rule \( l \rightarrow r \) of a CS-TRS \( (\mathcal{R}, \mu) \) is \( \mu \)-conservative if \( \mathcal{V} \text{ar}^\mu(r) \subseteq \mathcal{V} \text{ar}^\mu(l) \), i.e., all replacing variables in the right-hand-side are also replacing in the left-hand-side; \( (\mathcal{R}, \mu) \) is \( \mu \)-conservative if all its rules are (see [Luc06]).

**Proposition 3.1** Let \( \mathcal{R} = (C \uplus D, R) \) be a TRS and \( \mu \in M_\mathcal{R} \). If \( \mathcal{R} \) is \( \mu \)-conservative, then for all \( t \in M_{\infty, \mu} \), there exist \( l \rightarrow r \in R \), a substitution \( \sigma \) and a term \( s \) such that \( \text{root}(s) \in D \), \( r \supseteq^\mu s \), \( t \hookrightarrow^\mu \sigma(l) \rightarrow \sigma(r) \supseteq^\mu \sigma(s) \) and \( \sigma(s) \in M_{\infty, \mu} \).

**Proof.** Consider an infinite \( \mu \)-rewrite sequence starting from \( t \). By definition of \( M_{\infty, \mu} \), all proper \( \mu \)-replacing subterms of \( t \) are \( \mu \)-terminating. Therefore (since no \( \mu \)-rewriting step is possible on non-\( \mu \)-replacing subterms), \( t \) has an inner reduction to an instance \( \sigma(l) \) of the left-hand side of a rule \( l \rightarrow r \) of \( \mathcal{R} \): \( t \hookrightarrow^* \sigma(l) \rightarrow \sigma(r) \). Thus, we can write \( t = f(t_1, \ldots, t_k) \), \( \sigma(l) = f(l_1, \ldots, l_k) \).
for some $k$-ary defined symbol $f$, and $l_i \rightarrow^* \sigma(l_i)$ for all $i$, $1 \leq i \leq k$. Since all $l_i$ are $\mu$-terminating for $i \in \mu(f)$, the $\sigma(l_i)$ also are. By $\mu$-conservativity of $\mathcal{R}$, it follows that $\sigma(x)$ is $\mu$-terminating for every $x \in \text{Var}^\mu(r) \subseteq \text{Var}^\mu(l)$. Since $\sigma(r)$ is non-$\mu$-terminating, it contains a replacing subterm $t' \in \mathcal{M}_{\infty,\mu}$: $\sigma(r) \supseteq \mu t'$. Since variables $x \in \text{Var}^\mu(r)$ satisfy that $\sigma(x)$ is $\mu$-terminating, it follows that there is a nonvariable subterm $s$ of $r$ (i.e., $r \supseteq \mu s$) such that $\sigma(s) = t'$ as required. Thus, $\text{root}(s) \in \mathcal{D}$. □

**Definition 3.2** Let $\mathcal{R} = (\mathcal{F}, \mathcal{R}) = (\mathcal{C}, \mathcal{D}, \mathcal{R})$ be a TRS and $\mu \in M_\mathcal{R}$. Let

$$\text{DP}(\mathcal{R}, \mu) = \{l^f \rightarrow s^d \mid l \rightarrow r \in \mathcal{R}, r \supseteq \mu s, \text{root}(s) \in \mathcal{D}, l \not\supseteq \mu s\}$$

where $\mu^f(f) = \mu(f)$ if $f \in \mathcal{F}$ and $\mu^f(f^2) = \mu(f)$ if $f \in \mathcal{D}$.

Given a CS-TRS $(\mathcal{P}, \mu^d)$ of dependency pairs associated to a CS-TRS $(\mathcal{R}, \mu)$, an $(\mathcal{R}, \mathcal{P}, \mu^d)$-chain is a sequence of pairs $u_i \rightarrow v_i \in \mathcal{P}$ such that there is a substitution $\sigma$ satisfying $\sigma(u_i) \rightarrow^*_{\mathcal{R}, \mu^d} \sigma(v_{i+1})$ for $i \geq 1$. Here, as usual we assume that different occurrences of dependency pairs do not share any variable. We have the following.

**Theorem 3.3 (Completeness)** Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$. If $\mathcal{R}$ is $\mu$-terminating, then there is no infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^d)$-chain.

**Proof.** By contradiction. If there is an infinite $(\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^d)$-chain, then there is a substitution $\sigma$ and dependency pairs $u_i \rightarrow v_i \in \text{DP}(\mathcal{R}, \mu)$ for $i \geq 1$ such that

$$\sigma(u_1) \rightarrow^*_{\text{DP}(\mathcal{R}, \mu), \mu^d} \sigma(v_1) \rightarrow^*_{\mathcal{R}, \mu^d} \sigma(u_2) \rightarrow^*_{\text{DP}(\mathcal{R}, \mu), \mu^d} \sigma(v_2) \cdots$$

Assume that $u_i = f_i^{\sigma}(\overline{u}_i)$ and $v_i = g_i^{\sigma}(\overline{v}_i)$, and let $u'_i = f(\overline{u}_i)$ and $v'_i = g(\overline{v}_i)$. By definition of $\text{DP}(\mathcal{R}, \mu)$, $u'_i = l_i$ and $v'_i$ is a replacing subterm of the right-hand-side $r_i$ of a rule $l_i \rightarrow r_i$ in $\mathcal{R}$ (i.e., $r_i = C_i[v'_i]_{p_i}$ and $p_i \in \text{Pos}^\mu(r_i)$). Taking into account the definition of $\mu^d$, we obtain an infinite $\mu$-rewrite sequence

$$\sigma(u'_1) \rightarrow^*_{\mathcal{R}, \mu} \sigma(C_1[v'_1]_{p_1}) \rightarrow^*_{\mathcal{R}, \mu} \sigma(C_1[v'_2]_{p_1}) \rightarrow^*_{\mathcal{R}, \mu} \sigma(C_1[C_2[v'_2]_{p_2}]_{p_1}) \rightarrow^*_{\mathcal{R}, \mu} \cdots$$

which contradicts $\mu$-termination of $\mathcal{R}$. □

Note that Theorem 3.3 does not impose any restriction on the TRSs for guaranteeing completeness. Unfortunately, without $\mu$-conservativeness, the use of $\text{DP}(\mathcal{R}, \mu)$ does not guarantee the necessary correctness.

**Example 3.4** Consider the TRS $\mathcal{R}$:

$$a \rightarrow f(b, a) \quad f(b, y) \rightarrow f(y, c)$$

and $\mu(f) = \{1\}$. Then, $\text{DP}(\mathcal{R}, \mu)$ is:

$$A \rightarrow F(b, a) \quad F(b, y) \rightarrow F(y, c)$$

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(and \(\mu^*(F) = \{1\}\)) which does not induce any infinite chain of dependency pairs. However, \(\mathcal{R}\) is not \(\mu\)-terminating:

\[
a \leftarrow f(b, a) \leftarrow f(a, c) \leftarrow f(f(b, a), c) \leftarrow \ldots
\]

Note that \(\mathcal{R}\) is not \(\mu\)-conservative: \(\text{Var}^\mu(f(y, c)) - \text{Var}^\mu(f(b, y)) = \{y\}\).

**Theorem 3.5 (Restricted correctness)** Let \(\mathcal{R}\) be a TRS and \(\mu \in M_\mathcal{R}\). If \(\mathcal{R}\) is \(\mu\)-conservative and there is no infinite \((\mathcal{R}, \text{DP}(\mathcal{R}, \mu), \mu^*)\)-chain, then \(\mathcal{R}\) is \(\mu\)-terminating.

**Example 3.6** Consider the following TRS \(\mathcal{R}\):

\[
f(a) \rightarrow f(f(a)) \quad f(a) \rightarrow a
\]

and \(\mu(f) = \emptyset\). \(\text{DP}(\mathcal{R}, \mu)\) consists of the rule \(F(a) \rightarrow F(f(a))\) with \(\mu^*(F) = \emptyset\). Since \(\mu^*(F) = \emptyset\), no infinite chain is possible now and \(\mathcal{R}\) is proved to be \(\mu\)-terminating.

### 4 Context-sensitive dependency graph

As noticed by Arts and Giesl, the analysis of infinite sequences of dependency pairs can be made by looking at (the cycles \(\mathcal{C}\) of) the dependency graph associated to the TRS \(\mathcal{R}\). The nodes of the dependency graph are the dependency pairs in \(\text{DP}(\mathcal{R})\); there is an arc from a dependency pair \(u \rightarrow v\) to a dependency pair \(u' \rightarrow v'\) if there are substitutions \(\sigma\) and \(\theta\) such that \(\sigma(v) \rightarrow^*_R \theta(u')\). In the context-sensitive dependency graph, there is an arc from a dependency pair \(u \rightarrow v\) to a dependency pair \(u' \rightarrow v'\) if there are substitutions \(\sigma\) and \(\theta\) such that \(\sigma(v) \leftarrow^*_R, \mu^* \theta(u')\). Here, the use of \(\mu^*\), which restricts reductions on the arguments of the dependency pair symbols \(f^i\) is essential: given a set of dependency pairs associated to a CS-TRS \((\mathcal{R}, \mu)\), we have less arcs between them due to the presence of such replacement restrictions.

**Example 4.1** Consider the CS-TRS in Example 1.1. \(\text{DP}(\mathcal{R}, \mu)\) is:

\[
F(a, b, X) \rightarrow F(X, X, X)
\]

with \(\mu^*(F) = \{3\}\). Although the dependency graph contains a cycle (due to \(\sigma(F(X, X, X)) \rightarrow^* \sigma(F(a, b, Y))\) for \(\sigma(X) = \sigma(Y) = c\)), the CS-dependency graph contains no cycle because it is not possible to \(\mu^*\)-reduce \(\theta(F(X, X, X))\) into \(\theta(F(a, b, Y))\) for any substitution \(\theta\) (due to \(\mu^*(F) = \{3\}\)).

As noticed by Arts and Giesl, the presence of an infinite chain of dependency pairs correspond to a cycle in the dependency graph (but not vice-versa). Thus, as a corollary of Theorem 3.5, we have:

**Corollary 4.2** Let \(\mathcal{R}\) be a TRS and \(\mu \in M_\mathcal{R}\). If \(\mathcal{R}\) is \(\mu\)-conservative and the CS-dependency graph built from \(\text{DP}(\mathcal{R}, \mu)\) contains no cycle, then \(\mathcal{R}\) is \(\mu\)-terminating.

According to Corollary 4.2, and continuing Example 4.1, we conclude the \(\mu\)-termination of \(\mathcal{R}\) in Example 1.1. Of course, the interesting situation is
when the dependency graph contains cycles. The following definition borrows [HM04, Definition 8] and will be used later.

**Definition 4.3** Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_\mathcal{R}$. Let $\mathcal{C} \subseteq \text{DP}(\mathcal{R}, \mu)$ be a cycle. An infinite $\mu$-rewrite sequence in $\mathcal{R} \cup \mathcal{C}$ of the form

$$t_1 \xrightarrow*_{\mathcal{R}, \mu} s_1 \xrightarrow{\mathcal{C}, \mu} t_2 \xrightarrow*_{\mathcal{R}, \mu} s_2 \xrightarrow{\mathcal{C}, \mu} t_3 \xrightarrow*_{\mathcal{R}, \mu} \cdots$$

with $t_1 \in \mathcal{T}_{\text{sc}, \mu}$ is called $\mathcal{C}$-minimal if all rules in $\mathcal{C}$ are applied infinitely often.

Following Hirokawa and Middeldorp, proving $\mu$-termination boils down to proving the absence of $\mathcal{C}$-minimal $\mu$-rewrite sequences, for any cycle $\mathcal{C}$ in the CS-dependency graph.

### 4.1 Estimating the CS-dependency graph

In general, the dependency graph of a TRS is *not* computable and we need to use some approximation of it. Following [AG00], we describe how to approximate the CS-dependency graph of a CS-TRS $(\mathcal{R}, \mu)$. Let $\text{CAP}^\mu$ be given as follows: let $D$ be a set of defined symbols (in our context, $D = \mathcal{D} \cup \mathcal{D}^\mu$):

$$\text{CAP}^\mu(x) = x \text{ if } x \text{ is a variable}$$

$$\text{CAP}^\mu(f(t_1, \ldots, t_k)) = \begin{cases} y & \text{if } f \in D \\ f([t_1]^f, \ldots, [t_k]^f) & \text{otherwise} \end{cases}$$

where $y$ is intended to be a new, fresh variable which has not yet been used and given a term $s$, $[s]^f_i = \text{CAP}^\mu(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$. Let $\text{REN}^\mu$ given by: $\text{REN}^\mu(x) = y$ if $x$ is a variable and $\text{REN}^\mu(f(t_1, \ldots, t_k)) = f([t_1]^f, \ldots, [t_k]^f)$ for every $k$-ary symbol $f$, where given a term $s \in \mathcal{T}^\mu(\mathcal{F}, \mathcal{X})$, $[s]^f_i = \text{REN}^\mu(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$. Then, we have an arc from $u_i \rightarrow v_i$ to $u_j \rightarrow v_j$ if $\text{REN}^\mu(\text{CAP}^\mu(v_i))$ and $u_j$ unify; following [AG00], we say that $v_i$ and $u_j$ are $\mu$-connectable. The following result whose proof is similar to that of [AG00, Theorem 21] (we only need to take into account the replacement restrictions indicated by the replacement map $\mu$) formalizes the correctness of this approach.

**Proposition 4.4** Let $(\mathcal{R}, \mu)$ be a CS-TRS. If there is an arc from $u \rightarrow v$ to $u' \rightarrow v'$ in the CS-dependency graph, then $v$ and $u'$ are $\mu$-connectable.

**Example 4.5** (Ex. 4.1 continued) Since $\text{REN}^\mu(\text{CAP}^\mu(\text{F}(X, X, X))) = \text{F}(X, X, Z)$ and $\text{F}(a, b, Y)$ do not unify, we conclude that the CS-dependency graph for the CS-TRS $(\mathcal{R}, \mu)$ in Example 1.1 contains no cycles.

### 5 Subterm criterion

In [HM04], Hirokawa and Middeldorp introduce a very interesting *subterm criterion* which permits to ignore certain cycles of the dependency graph.
Definition 5.1 [HM04] Let \( \mathcal{R} \) be a TRS and \( \mathcal{C} \subseteq \text{DP}(\mathcal{R}) \) such that every dependency pair symbol in \( \mathcal{C} \) has positive arity. A simple projection for \( \mathcal{C} \) is a mapping \( \pi \) that assigns to every \( k \)-ary dependency pair symbol \( f^k \) in \( \mathcal{C} \) an argument position \( i \in \{1, \ldots, k\} \). The mapping that assigns to every term \( f^k(t_1, \ldots, t_k) \in \mathcal{T}^k(\mathcal{F}, \mathcal{X}) \), with \( f^k \) a dependency pair symbol in \( \mathcal{R} \), its argument position \( \pi(f^k) \) is also denoted by \( \pi \).

In the following result, for a simple projection \( \pi \) and \( \mathcal{C} \subseteq \text{DP}(\mathcal{R}, \mu) \), we let \( \pi(\mathcal{C}) = \{ \pi(u \to v) \mid u \to v \in \mathcal{C} \} \), where \( \pi(u \to v) = \pi(u) \to \pi(v) \) if \( u \to v \) is a dependency pair. Note that \( u, v \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \), but \( \pi(u), \pi(v) \in \mathcal{T}(\mathcal{F}, \mathcal{X}) \).

Theorem 5.2 Let \( \mathcal{R} \) be a TRS and \( \mu \in M_{\mathcal{R}} \). Let \( \mathcal{C} \) be a cycle in \( \text{DG}(\mathcal{R}) \). If there exists a simple projection \( \pi \) for \( \mathcal{C} \) such that \( \pi(\mathcal{C}) \subseteq \geq_{\mu} \) and \( \pi(\mathcal{C}) \cap \triangleright_{\mu} \neq \emptyset \), then there are no minimal \( \mathcal{C} \mu^n \)-rewrite sequences.

Proof. The proof is completely analogous to that of [HM04, Theorem 11]. The only difference is that we need to deal with the \( \mu \)-subterm relation \( \geq_{\mu} \); this is because, we need to use the following commutation property: \( \triangleright_{\mu} \circ \leq_{\mu} \subseteq \leq_{\mu} \circ \triangleright_{\mu} \) which does not hold if \( \triangleright \) is used instead. \( \square \)

Example 5.3 Consider the CS-TRS \( (\mathcal{R}, \mu) \) in Example 1.2. The context-sensitive dependency graph consists of seven pairs with only four cycles:

(C1) \{SQR(s(X)) \to SQR(X)\}
(C2) \{DBL(s(X)) \to DBL(X)\}
(C3) \{ADD(s(X), Y) \to ADD(X, Y)\}
(C4) \{HALF(s(s(X))) \to HALF(X)\}

Each of them can be removed by using the subterm criterion. Thus, by Theorems 3.5 and 5.2, the system is \( \mu \)-terminating.

As remarked by Hirokawa and Middeldorp, the practical use of Theorem 5.2 concerns the so-called strongly connected components of the dependency graph, rather than the cycles themselves (which are exponentially many) [HM04, HM05].

For the cycles left after the removal performed by means of the subterm criterion, the absence of infinite chains is checked by finding, for those cycles, a reduction pair \( (\geq, \sqsubseteq) \). In our setting, we can relax the monotonicity requirements on the first component of the reduction pair \( (\geq, \sqsubseteq) \). We will use \( \mu \)-reduction pairs \( (\geq_{\mu}, \sqsubseteq) \) where \( \geq_{\mu} \) is a stable and \( \mu \)-monotonic preordering which is compatible with the well-founded and stable ordering \( \sqsubseteq \), i.e., \( \geq_{\mu} \circ \sqsubseteq \subseteq \sqsubseteq \) or \( \sqsubseteq \circ \geq_{\mu} \subseteq \sqsubseteq \). On the other hand, the use of argument filterings, which is standard in the current formulations of the dependency pairs method, also adapts without changes to the context-sensitive setting. Here, an argument filtering \( \pi \) for a signature \( \mathcal{F} \) is a mapping that assigns to every \( k \)-ary function symbol \( f \in \mathcal{F} \) an argument position \( i \in \{1, \ldots, k\} \) or a (possibly empty) list \( [i_1, \ldots, i_m] \) of argument positions with \( 1 \leq i_1 < \cdots < i_m \leq k \).

The signature \( \mathcal{F}_\pi \) consists of all function symbols \( f \) such that \( \pi(f) \) is some list
\[ [i_1, \ldots, i_m], \text{ where in } \mathcal{F}_\pi, \text{ the arity of } f \text{ is } m. \] Every argument filtering induces a mapping from \( T(\mathcal{F}, \mathcal{A}) \) to \( T(\mathcal{F}_\pi, \mathcal{A}) \), also denoted by \( \pi \):

\[
\pi(t) = \begin{cases} 
  t & \text{if } t \text{ is a variable} \\
  \pi(t_i) & \text{if } t = f(t_1, \ldots, t_k) \text{ and } \pi(f) = i \\
  f(\pi(t_{i_1}), \ldots, \pi(t_{i_m})) & \text{if } t = f(t_1, \ldots, t_k) \text{ and } \pi(f) = [i_1, \ldots, i_m]
\end{cases}
\]

The proof of the following Theorem is completely analogous to standard ones (see, e.g., [HM04, Theorem 18]).

**Theorem 5.4 (Use of the CS-dependency graph)** Let \( \mathcal{R} \) be a TRS and \( \mu \in M_{\mathcal{R}} \). Let \( \mathcal{C} \) be a cycle in the CS-dependency graph. If there is an argument filtering \( \pi \) and an \( \mu \)-reduction pair \((\succeq_{\mu}, \sqsubseteq)\) such that \( \pi(R) \subseteq \succeq_{\mu}, \pi(C) \subseteq (\succeq_{\mu} \cup \sqsubseteq) \) and \( \pi(C) \cap \sqsubseteq \neq \emptyset \), then there is no \( \mathcal{C} \)-minimal \( \mu \)-rewrite sequence.

**Example 5.5** Consider the TRS \( \mathcal{R} \):

\[
g(X) \rightarrow h(X) \quad h(d) \rightarrow g(c) \quad c \rightarrow d
\]

together with \( \mu(g) = \mu(h) = \emptyset \) [Zan97, Example 1]. Then, DP(\( \mathcal{R}, \mu \)) is:

\[
G(X) \rightarrow H(X) \quad H(d) \rightarrow G(c)
\]

Note that the dependency graph contains a single cycle including both of them. We can prove the \( \mu \)-termination of \( \mathcal{R} \) by using the following polynomial interpretation over the natural numbers:

\[
[g](x) = 0 \quad [c] = 1 \quad [G](x) = x^2 - 3x + 4 \\
h(x) = 0 \quad [d] = 0 \quad [H](x) = x^2 - 3x + 3
\]

which induces a \( \mu \)-reduction pair \((\succeq, >)\) which is compatible with \( \mathcal{R} \) and the cycle in DP(\( \mathcal{R}, \mu \)) in the sense of Theorem 5.4. By Theorem 3.5, \( \mathcal{R} \) is \( \mu \)-terminating.

### 6 Conclusions

We have shown how to use dependency pairs in proofs of termination of CSR. Regarding the practical use of the CS-dependency pairs in proofs of termination of CSR, we have shown how to build and use the corresponding CS-dependency graph to either prove that there are cycles which do not need to be considered at all (Theorem 5.2) or to prove that the rules of the TRS and the cycles in the CS-dependency graph are compatible with some reduction pair (Theorem 5.4). We have implemented these ideas as part of the termination tool mu-term [AGIL06].
References


