

From Matrix Interpretations over the Rationals to Matrix Interpretations over the Naturals

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Abstract. Matrix interpretations generalize linear polynomial interpretations and have been proved useful in the implementation of tools for automatically proving termination of Term Rewriting Systems. In view of the successful use of rational coefficients in polynomial interpretations, we have recently generalized traditional matrix interpretations (using *natural* numbers in the matrix entries) to incorporate *real* numbers. However, existing results which formally prove that polynomials over the reals are *more powerful* than polynomials over the naturals for proving termination of rewrite systems *failed* to be extended to matrix interpretations. In this paper we get deeper into this problem. We show that, under some conditions, it is possible to transform a matrix interpretation over the rationals satisfying a set of symbolic constraints into a matrix interpretation over the naturals (using *bigger* matrices) which still satisfies the constraints.

Keywords: Matrix and Polynomial Interpretations, Program Analysis, Termination.

1 Introduction

Constraint solving is an essential technique for the implementation of automatic verification systems. Many verification problems can be expressed as sets of symbolic constraints which have to be tested for satisfaction or even solved to give some explicit solution certifying their satisfaction. For instance, termination problems are often expressed as conjunctions of *weak* or *strict symbolic constraints* like $e \succeq e'$ or $e \succ e'$ between expressions e and e' coming from (parts of) the programs [6]. Automatic termination tools have to *check* these constraints and eventually provide an appropriate certificate. A standard approach is using algebraic interpretations which translate the *symbolic* constraints into some kind of *arithmetic* constraints.

Example 1. Consider the following Term Rewriting System (TRS) \mathcal{R} [1,2]:

$$f(f(X)) \rightarrow f(g(f(X))) \tag{1}$$

$$f(g(f(X))) \rightarrow X \tag{2}$$

A proof of termination with *dependency pairs* can be easily obtained as follows [3]: Consider the following rules (called *dependency pairs*) associated to \mathcal{R} :

$$F(f(X)) \rightarrow F(g(f(X))) \tag{3}$$

$$F(f(X)) \rightarrow F(X) \tag{4}$$

The following polynomial interpretation with *rational* coefficients

$$[f](x) = 2x + 2 \quad [g](x) = \frac{1}{2}x + \frac{1}{2} \quad [F](x) = x$$

can be used to prove termination of \mathcal{R} by showing that $[l] \geq [r]$ for the rules (1), (2) (where $[l]$ and $[r]$ are the interpretations of terms l and r which is obtained by structural induction), and $[u] > [v]$ for the dependency pairs (3), (4).

A recent and fruitful approach is using *matrix interpretations* [10], where the k -ary symbols f are given *parametric* matrix functions $[f]$, e.g., $F_1x_1 + \dots + F_kx_k + F_0$, where the F_i 's are (square) matrices of some fixed dimension n and F_0 is an n -tuple. The variables x_1, \dots, x_k are intended to range on n -tuples as well. In [10], only natural numbers are used both in matrices and n -tuples.

Example 2. The following matrix interpretation *over the naturals*

$$[f](x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad [g](x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \quad [F](x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x$$

can also be used for proving termination of \mathcal{R} in Example 1. Both interpretations have been automatically obtained by using MU-TERM [13].

In [1,2], Endrullis et al.'s framework was extended to matrices containing *real numbers* in the entries. The adaptation was motivated by a number of recent theoretical works and experimental evaluations showing that *polynomial* interpretations over the rationals can be advantageously used instead of polynomial interpretations over the naturals [5,11,14]. In [11], syntactic conditions ensuring that, when dealing with *linear* polynomial interpretations, real coefficients *must be* used for addressing the corresponding termination problem, were given for the first time. The extension of these results to matrix interpretations over the reals failed [1,2].

Thus, the following question arises: Are rational numbers somehow *unnecessary* when dealing with matrix interpretations? By examining the proofs of termination contributed to the International Competition on Termination¹ by tools like **Jambox** [9], which makes extensive use of matrix interpretations over the naturals, and comparing them to the corresponding ones generated by tools like MU-TERM, which emphasizes the use of polynomials over the rationals [15], one may notice that, in many cases, proofs of termination with polynomials over the rationals somehow correspond to proofs using matrix interpretations whose matrices have specific shapes. Examples 1 and 2 illustrate this connection, which we substantiate in this paper: the bigger are the *values* of the (possibly rational) coefficients in the polynomial interpretation the more non-null entries are in the

¹ See <http://termcomp.uibk.ac.at/termcomp/>

corresponding matrix coefficients. In this paper we investigate this phenomenon. In Section 2 we develop a notion of *numeric matrix representation* which permits a representation of a natural number as a matrix of (smaller) natural numbers. This representation preserves the usual arithmetics of natural numbers (addition and product). Therefore, we can think of such matrices as having a *value*; the value of the number from which they were obtained. Then, in Section 3 we show how to extend this process to transform a matrix of natural numbers into a (bigger) matrix of (smaller) natural numbers. As a consequence, we prove that every matrix of natural numbers can be represented as a *bit* matrix with entries in $\{0, 1\}$ which still preserves its ‘value’ and arithmetic behavior. In Section 4, we address the problem of representing arbitrary *rational numbers* as matrices of integer numbers. We argue that this is possible only for finite subsets of rational numbers. We investigate the use of *nilpotent* matrices (i.e., square matrices which become null after a finite number of selfproducts) as suitable devices for achieving this. In Section 5, we introduce a new generalization of matrix interpretations, which we call *block-based matrix interpretations*. Essentially, we view a matrix as structured into blocks of (sums of) *constant* or *scalar* matrices. In Section 6, we investigate how the satisfaction of different kind of constraints is *preserved* under the matrix transformations investigated in the previous sections. As a consequence of our results, we prove that TRSs which can be proved terminating by using matrix interpretations over the naturals can also be proved terminating by using a matrix interpretation based on *bit* matrices. Furthermore, we show that, under some conditions, proofs of termination which are carried out by polynomial or matrix interpretations over the rationals can also be obtained by using matrix interpretations over the naturals (like in Example 2). Section 7 summarizes our contribution and concludes.

2 Numbers as Matrices

In the following, we use the standard notations and terminology for matrices [12,17,18]. Given $p, q \in \mathbb{N}_{>0}$ and a set of numbers N (usually \mathbb{R} , \mathbb{Q} , or \mathbb{N}), we write $A \in N^{p \times q}$ to say that A is a matrix of p rows and q columns with entries $A_{ij} \in N$ (a $p \times q$ -matrix for short). If $p = q$, then A is a *square* matrix.

We investigate the representation of (rational) numbers as matrices of integer numbers satisfying some structural properties. Of course, we want to ensure that the representation preserves (part of) the algebraic structure of the considered numeric domain, in such a way that, for instance, the arithmetic operations and orderings among numbers (of the considered kind) can be implemented by using matrix operations and orderings.

We view a rational number as a product $p\frac{1}{q}$ for an integer number $p \in \mathbb{Z}$ and a positive natural number $q \in \mathbb{N}_{>0}$. First, we formalize a generic framework for representing real numbers as matrices: mappings $\mu : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ and $\rho : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ provide a representation of real numbers as matrices and vice versa.

Definition 1 (Numeric matrix representation). *Let $N \subseteq \mathbb{R}$ be a subset of real numbers, $p, q \in \mathbb{N}_{>0}$, and $M \subseteq \mathbb{R}^{p \times q}$ be a subset of matrices. A numeric*

matrix representation is a pair (μ, ρ) of mappings $\mu : N \rightarrow M$ (called a numeric representation) and $\rho : M \rightarrow N$ (called a matrix valuation) such that $\rho \circ \mu = id$.

Let $\mathbf{1}_{p \times q}$ be the $p \times q$ -matrix all whose entries contain 1. A matrix $C = c\mathbf{1}_{p \times q}$ for some $c \in \mathbb{R}$ (i.e., whose entries are settled to c) is called a *constant* matrix. The identity (square) matrix of size n is denoted I_n . A matrix $S = cI_n$ for some $c \in \mathbb{R}$ is called a *scalar* matrix. We consider the following numeric matrix representation.

Definition 2. Let $m, p, q \in \mathbb{N}_{>0}$ and $A \in \mathbb{R}^{p \times q}$. We let

1. $\mu_m^{p \times q}(x) = \frac{mx}{pq} \mathbf{1}_{p \times q}$, i.e., each real number x is mapped to a constant $p \times q$ -matrix with entries $\frac{mx}{pq}$.
2. $\rho_m(A) = \frac{\sum_{i=1}^p \sum_{j=1}^q A_{ij}}{m}$, i.e., each matrix A is mapped to the number which is obtained by adding all entries in A and then dividing this number by m .

In the following, we prove some properties of ρ_m which are used below.

Proposition 1. Let $m, p, q \in \mathbb{N}_{>0}$ and $A, B \in \mathbb{R}^{p \times q}$. Then, $\rho_m(A + B) = \rho_m(A) + \rho_m(B)$ and $\rho_m(\alpha A) = \alpha \rho_m(A)$ for all $\alpha \in \mathbb{R}$.

Proposition 2. Let $p, q, r \in \mathbb{N}_{>0}$, $A \in \mathbb{R}^{p \times q}$ and $B \in \mathbb{R}^{q \times r}$. If B (resp. A) is scalar and $q \leq r$ (resp. $q \leq p$), or B (resp. A) is a constant matrix, then $\rho_p(AB) = \rho_p(A)\rho_q(B)$.

Propositions 1 and 2 entail the following.

Corollary 1. Let $p, q, r \in \mathbb{N}_{>0}$, $A \in \mathbb{R}^{p \times q}$ and $B \in \mathbb{R}^{q \times r}$. If B (resp. A) is an additive combination of scalar and constant matrices, then $\rho_p(AB) = \rho_p(A)\rho_q(B)$.

Corollary 2. Let $n \in \mathbb{N}_{>0}$ and A, B be n -square matrices. If B is an additive combination of scalar or constant matrices, then $\rho_n(AB) = \rho_n(A)\rho_n(B) = \rho_n(BA)$.

If we consider only n -square matrices for representations, then $\mu'_n(x) = xI_n$ could also be used with ρ_n as a numeric matrix representation.

Remark 1 (Use of vectors). Since vectors \mathbf{v} can be seen as special matrices $\mathbf{v} \in \mathbb{R}^{n \times 1}$ of n rows and a single column, the numeric matrix representation in Definition 2 can also be used to represent real numbers as vectors. In particular, we get $\mu_n^{n \times 1}(x) = x\mathbf{1}_n$ and $\rho_n(\mathbf{v}) = \frac{\sum_{i=1}^n v_i}{n}$.

3 Transforming Matrices of Numbers

We can *extend* any numeric representation $\mu : \mathbb{R} \rightarrow \mathbb{R}^{p \times q}$ to a mapping $\mu : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{mp \times nq}$ from $m \times n$ -matrices A into $mp \times nq$ -matrices $\mu(A)$ by just replacing the numeric entries A_{ij} in A by the corresponding matrices $\mu(A_{ij})$, i.e., $\mu(A) = (\mu(A_{ij}))_{i=1, j=1}^{i=m, j=n}$. The new matrix can be viewed as a *block* matrix whose blocks are $\mu(A_{ij})$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 3. We can transform the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ by using $\mu_m^{p \times q}$ in Definition 2; with $m = p = q = 3$, we obtain:

$$\mu_3^{3 \times 3} \left(\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) = \begin{pmatrix} \mu_3^{3 \times 3}(3) & \mu_3^{3 \times 3}(0) \\ \mu_3^{3 \times 3}(0) & \mu_3^{3 \times 3}(3) \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 3} \\ \mathbf{0}_{3 \times 3} & \mathbf{1}_{3 \times 3} \end{pmatrix}$$

The interesting feature of the matrix obtained in Example 3 is that it is a *bit* matrix whose entries are either² 0 or 1. Note that this is due to the use of $\mu_3^{3 \times 3}$ which permits a representation of ‘3’ as a constant matrix $\mathbf{1}_{3 \times 3}$ with 1’s only. For the numeric matrix representation in Definition 2, we have the following.

Proposition 3. *Let $m, n, p, q, r, s \in \mathbb{N}_{>0}$ and $A \in \mathbb{R}^{r \times s}$. Then, $\rho_{mn}(\mu_m^{p \times q}(A)) = \rho_n(A)$.*

Theorem 1. *Let $m, n, p, q, r, s, t, u \in \mathbb{N}_{>0}$. If $A, B \in \mathbb{R}^{r \times s}$, then, $\rho_m(A) + \rho_m(B) = \rho_{mn}(\mu_n^{p \times q}(A) + \mu_n^{p \times q}(B))$ and $\rho_m(\alpha A) = \alpha \rho_{mn}(\mu_n^{p \times q}(A))$ for all $\alpha \in \mathbb{R}$. If $A \in \mathbb{R}^{s \times t}$ and $B \in \mathbb{R}^{t \times u}$, then $\rho_m(AB) = \rho_{mn}(\mu_n^{p \times q}(A) \mu_n^{q \times r}(B))$.*

Propositions 1 and 2 entail the following.

Corollary 3. *Let $m, n, p, q, r, s \in \mathbb{N}_{>0}$, $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times s}$. If A is an additive combination of scalar and constant matrices, then $\rho_m(AB) = \rho_{mn}(\mu_n^{p \times q}(A) \mu_n^{q \times r}(B)) = \rho_{mn}(\mu_n^{p \times q}(A)) \rho_{mn}(\mu_n^{q \times r}(B))$.*

3.1 Representing Integer Numbers as Matrices

In the following, we use $\mu_n^{p \times q}$ in Definition 2 as a basis for the following representation mapping for *integer numbers*.

Definition 3 (Representing integer numbers as matrices). *Let $n \in \mathbb{N}$ be such that $n > 1$ and μ_n be given as follows: for all $x \in \mathbb{Z}$,*

$$\mu_n(x) = \begin{cases} \frac{x}{n} \mathbf{1}_{n \times n} & \text{if } n \text{ divides } x \\ x I_n & \text{otherwise} \end{cases}$$

We also define $\nu_n(x) = x \mathbf{1}_n$ to represent a number x as a n -dimensional vector.

Example 4. The matrix $\mu_2(4) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ represents 4 according to Definition 3.

Note that, for all $n \in \mathbb{N}$, (μ_n, ρ_n) (with ρ_n as in Definition 2) is a numeric matrix representation for integer numbers. We obtain the following:

Proposition 4 (Decrease of natural entries in matrices). *Every matrix $A \in \mathbb{N}^{p \times q}$ such that $n = \max(A) > 1$ can be represented as a matrix $A' \in \mathbb{N}^{np \times nq}$ such that, for all $m \in \mathbb{N}_{>0}$, $\rho_m(A) = \rho_{mn}(A')$ and $\max(A) > \max(A')$.*

Obviously, Proposition 4 entails the following.

Corollary 4 (Natural matrices as bit matrices). *Every matrix $A \in \mathbb{N}^{p \times q}$ can be represented as a bit matrix $A' \in \{0, 1\}^{np \times nq}$ for some $n \in \mathbb{N}_{>0}$ and for all $m \in \mathbb{N}_{>0}$, $\rho_m(A) = \rho_{mn}(A')$.*

² Matrices with entries in $\{0, 1\}$ are called $(0, 1)$ -matrices in [18, Section 8.2].

4 Representation of Rational Numbers Below 1

In this section, we investigate matrix representations (over the naturals) which can be used to deal with rational numbers $\frac{1}{q}$ for some $q \in \mathbb{N}_{>0}$. A q -square matrix $A_{\frac{1}{q}}$ which is almost null except for a single entry of value 1 can be used to represent $\frac{1}{q}$ because $\rho_q(A_{\frac{1}{q}}) = \frac{1}{q}$. By Corollary 2, for $A_p = pI_q$ we get $\rho_q(A_p A_{\frac{1}{q}}) = \rho_q(A_{\frac{1}{q}} A_p) = \rho_q(A_p) \rho_q(A_{\frac{1}{q}}) = \frac{p}{q}$. Therefore, we can represent a rational number $\frac{p}{q}$ as a q -square matrix with a single entry of value p . However, we have to change the size of the matrix if a different number $\frac{p'}{q'}$ with $q \neq q'$ is considered.

Remark 2. Note that there is no generic representation of all rational numbers by using matrices over the naturals of a given dimension which is able to represent their values and appropriate comparisons among them. Such a representation should be able to represent a decreasing sequence $1 > \frac{1}{2} > \frac{1}{3} > \dots > \frac{1}{n} > \dots$ by means of matrices $A_{\frac{1}{n}} \in \mathbb{N}^{p \times q}$ for all $n \in \mathbb{N}_{>0}$ satisfying $\rho_m(A_{\frac{1}{n}}) > \rho_m(A_{\frac{1}{n+1}})$ for all $n \in \mathbb{N}_{>0}$. Equivalently, we should have $m\rho_m(A_{\frac{1}{n}}) > m\rho_m(A_{\frac{1}{n+1}})$ for all $n \in \mathbb{N}_{>0}$. Since $m\rho_m(A_{\frac{1}{n}}), m\rho_m(A_{\frac{1}{n+1}}) \in \mathbb{N}$, this would imply the existence of an infinite decreasing sequence of natural numbers, which is not possible.

Furthermore, the product of rational numbers $\frac{p}{q}$ and $\frac{p'}{q'}$ represented as the q -square matrices indicated above is *not* preserved by the matrix product. Thus, in the following, we look for better representations.

4.1 Use of Nilpotent Matrices

Nilpotent matrices are n -square matrices B satisfying $B^k = \mathbf{0}_{n \times n}$ for some positive integer $k \in \mathbb{N}_{>0}$ [18, Section 4.1] (which is called the *degree of nilpotency* [12, page 12] or the *index* of nilpotency of B [17, page 396]). The degree of nilpotency k of a n -square matrix A is bounded by n : $k \leq n$ [17, Exercise 7.7.1]. Given n , the following n -square matrix (called a *Jordan block* [17, Page 579]):

$$J_n = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

(i.e., there are $n - 1$ ones in the superdiagonal of J_n and all other entries in J_n are zeroes) is nilpotent of degree n , i.e., $J_n^n = \mathbf{0}_{n \times n}$.

Write $J_n = [0, Z_1, \dots, Z_{n-1}]$, where Z_i is an almost null vector containing a single 1 in the i -th entry. For instance, $Z_1 = (1, 0, \dots, 0)^T$. Then, it is not difficult to see that J_n^p is obtained by introducing $p - 1$ columns of zeros from the left side of the matrix and shifting columns Z_i to the right just throwing those which exceed the n -th position, i.e., $J_n^p = [0, \dots, 0, Z_1, \dots, Z_{n-p}]$.

Remark 3. J_n Is also known as a *shift* matrix because, for an arbitrary matrix A , $J_n A$ is obtained by shifting the rows of A upwards by one position and

introducing a new bottom row of zeroes. Similarly AJ_n shifts the columns of A to the right and introduces a new leftmost column of zeroes.

Note that, for all $p \in \mathbb{N}$,

$$\rho_m(J_n^p) = \begin{cases} \frac{n-p}{m} & \text{if } p < n \\ 0 & \text{if } p \geq n \end{cases}$$

The following result is obvious.

Proposition 5. *Let $m, n \in \mathbb{N}_{>0}$ and $0 \leq p < q \leq n$. Then, $\rho_m(J_n^p) = \rho_m((J_n^T)^p) > \rho_m((J_n^T)^q) = \rho_m(J_n^q)$. For all $A \in \mathbb{R}^{n \times r}$, $\rho_m(J_n^p A) \geq \rho_m(J_n^q A)$. For all $A \in \mathbb{R}^{r \times n}$, $\rho_m(AJ_n^p) \geq \rho_m(AJ_n^q)$.*

In general it is *not* true that $\rho_m(J^p A) = \rho_m(AJ^p)$ for square matrices A . Due to Proposition 5 and Corollary 1, for additive combinations A of *constant* and *scalar* matrices we have the following:

Corollary 5. *Let $m, n \in \mathbb{N}_{>0}$ and $0 \leq p < q \leq n$. If $A \in \mathbb{R}^{n \times r}$ (resp. $A \in \mathbb{R}^{r \times n}$) is an additive combination of constant and scalar matrices, then $\rho_m(J_n^p A) = \rho_m(J_n^p) \rho_m(A) > \rho_m(J_n^q) \rho_m(A) = \rho_m(J_n^q A)$ (resp. $\rho_m(AJ_n^p) = \rho_m(A) \rho_m(J_n^p) > \rho_m(A) \rho_m(J_n^q) = \rho_m(AJ_n^q)$).*

Thus, the following property of rational numbers $r = \frac{1}{n}$ for some $n \in \mathbb{N}_{>0}$ is simulated by the representation of integer numbers as (additive combinations of) constant or scalar matrices, and rational numbers $\frac{1}{n}$ as powers of Jordan blocks: for all $n \in \mathbb{N}$ and positive rational number r , $0 < r < 1$, we have $n > nr > nr^2 > \dots$.

Example 5. We can use J_2 to represent $\frac{1}{2}$: $\rho_2(J_2) = \frac{1}{2}$. However, J_2^2 , which is expected to correspond to $\frac{1}{4}$ does *not* fit this expectation: $\rho_2(J_2^2) = \rho_2(\mathbf{0}_2) = 0$.

Theorem 2. *If A_1, \dots, A_N are n -square matrices such that, for all i , $1 \leq i \leq N$, either*

1. A_i is scalar or constant, or
2. $A_i = J_n^{p_i}$ for some $p_i \in \mathbb{N}$ and then both A_{i-1} (if $i > 1$) and A_{i+1} (if $i < N$) are constant matrices,

then $\rho_n(\prod_{i=1}^N A_i) = \prod_{i=1}^N \rho_n(A_i)$.

As remarked above, it is *not* possible to use matrices of natural numbers of a fixed size n to represent *all* fractions $\frac{1}{q}$ for $q \in \mathbb{N}_{>0}$. Instead, we will consider the problem of representing *finite subsets* $\mathcal{Q} \subseteq \{\frac{1}{q} \mid q \in \mathbb{N}_{>0}\}$ by using n -square (nilpotent) matrices in such a way that the following property is fulfilled by the representation (μ, ρ) : for all $x, y \in \mathcal{Q}$ such that $xy \in \mathcal{Q}$, $\rho(\mu(x)\mu(y)) = xy = \rho(\mu(x))\rho(\mu(y))$, i.e., the number $\rho(\mu(x)\mu(y))$ which corresponds to the matrix product $\mu(x)\mu(y)$ of matrices $\mu(x)$ and $\mu(y)$ representing $x \in \mathcal{Q}$ and $y \in \mathcal{Q}$ is exactly $xy \in \mathcal{Q}$. In Section 6 we discuss how to take benefit from this.

The dimension n of the matrices involved in the representation of \mathcal{Q} is determined by the *least* element in \mathcal{Q} . For instance, we can fix n such that $\frac{1}{n}$ is

the least number in \mathcal{Q} . Then, an obvious representative for $\frac{1}{n}$ is J_n^{n-1} because $\rho_n(J_n^{n-1}) = \frac{1}{n}$ (see Example 5). However, the feasibility of this simple approach depends on the other values in \mathcal{Q} .

Example 6. Let $\mathcal{Q} = \{\frac{1}{2}, \frac{1}{4}\}$. If we fix J_4^3 to be the representation of $\frac{1}{4}$, then the representation of $\frac{1}{2}$ should be J_4^2 (because $\rho_4(J_4^2) = \frac{1}{2}$). However, $\rho_4((J_4^2)^2)$ is *not* $\frac{1}{4}$ as one could expect; instead, $\rho_4((J_4^2)^2) = 0$. The following *block* matrices of size 4 whose blocks are combinations of (transposed) Jordan blocks can be used to represent \mathcal{Q} as required:

$$Q_{\frac{1}{2}} = \begin{pmatrix} J_2 & J_2^T \\ \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix} \quad Q_{\frac{1}{4}} = \begin{pmatrix} \mathbf{0}_{2 \times 2} & J_2 J_2^T \\ \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix}$$

Note that $\rho_4(Q_{\frac{1}{2}}) = \frac{1+1}{4} = \frac{1}{2}$, $\rho_4(Q_{\frac{1}{4}}) = \frac{0+1}{4} = \frac{1}{4}$, and $Q_{\frac{1}{2}} Q_{\frac{1}{2}} = Q_{\frac{1}{4}}$.

Example 7. Let $\mathcal{Q} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$. The following matrices of size 8:

$$Q_{\frac{1}{2}} = \begin{pmatrix} J_4 & J_4^3 \\ \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} \end{pmatrix} \quad Q_{\frac{1}{4}} = \begin{pmatrix} J_4^2 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} \end{pmatrix} \quad Q_{\frac{1}{8}} = \begin{pmatrix} J_4^3 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \mathbf{0}_{4 \times 4} \end{pmatrix}$$

can be used to represent \mathcal{Q} . Note that $\rho_8(Q_{\frac{1}{2}}) = \frac{3+1}{8} = \frac{1}{2}$, $\rho_8(Q_{\frac{1}{4}}) = \frac{2+0}{8} = \frac{1}{4}$, $\rho_8(Q_{\frac{1}{8}}) = \frac{1+0}{8} = \frac{1}{8}$, $Q_{\frac{1}{2}} Q_{\frac{1}{2}} = Q_{\frac{1}{4}}$ and $Q_{\frac{1}{2}} Q_{\frac{1}{4}} = Q_{\frac{1}{8}}$, as required.

Example 8. Let $\mathcal{Q} = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$. The following (block) matrices of size 6 can be used to represent \mathcal{Q} :

$$Q_{\frac{1}{2}} = \begin{pmatrix} J_3 & (J_3^2)^T \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{pmatrix} \quad Q_{\frac{1}{3}} = \begin{pmatrix} J_3^2 & J_3 (J_3^2)^T \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{pmatrix} \quad Q_{\frac{1}{6}} = \begin{pmatrix} \mathbf{0}_{3 \times 3} & J_3^2 (J_3^2)^T \\ \mathbf{0}_{3 \times 3} & \mathbf{0}_{3 \times 3} \end{pmatrix}$$

Note that $\rho_6(Q_{\frac{1}{2}}) = \frac{2+1}{6} = \frac{1}{2}$, $\rho_6(Q_{\frac{1}{3}}) = \frac{1+1}{6} = \frac{1}{3}$, and $\rho_6(Q_{\frac{1}{6}}) = \frac{0+1}{6} = \frac{1}{6}$. Again, it is not difficult to see that $Q_{\frac{1}{2}} Q_{\frac{1}{3}} = Q_{\frac{1}{6}}$. Note, however, that if we add $\frac{1}{4}$ to \mathcal{Q} , then we could *not* use these matrices for representing $\frac{1}{4}$. In particular, $Q_{\frac{1}{2}}^2 = Q_{\frac{1}{3}}$, i.e., $\rho_6(Q_{\frac{1}{2}}^2) \neq \frac{1}{4}$ as should be the case.

5 Matrix Interpretations Revisited

As remarked above, termination problems in term rewriting are usually translated into conjunctions of *weak* or *strict symbolic constraints* like $s \succeq t$ or $s \succ t$ between terms s and t coming from (parts of) the TRS. In order to check the satisfaction of these constraints, we need to use *term (quasi-)orderings*. Such term orderings can be obtained by giving appropriate *interpretations* to the function symbols of a signature. Given a signature \mathcal{F} , an \mathcal{F} -algebra is a pair $\mathcal{A} = (\mathbf{A}, \mathcal{F}_{\mathbf{A}})$, where \mathbf{A} is a set and $\mathcal{F}_{\mathbf{A}}$ is a set of mappings $f_{\mathbf{A}} : \mathbf{A}^k \rightarrow \mathbf{A}$ for each $f \in \mathcal{F}$ where $k = ar(f)$. For a given valuation mapping $\alpha : \mathcal{X} \rightarrow \mathbf{A}$, the evaluation mapping $[\alpha] : \mathcal{T}(\mathcal{F}, \mathcal{X}) \rightarrow \mathbf{A}$ is inductively defined by $[\alpha](x) = \alpha(x)$ if $x \in \mathcal{X}$ and $[\alpha](f(t_1, \dots, t_k)) = f_{\mathbf{A}}([\alpha](t_1), \dots, [\alpha](t_k))$ for $x \in \mathcal{X}$, $f \in \mathcal{F}$,

$t_1, \dots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Given a term t with $\mathcal{V}ar(t) = \{x_1, \dots, x_n\}$, we write $[t]$ to denote the *function* $F_t : \mathbf{A}^n \rightarrow \mathbf{A}$ given by $F_t(a_1, \dots, a_n) = [\alpha_{(a_1, \dots, a_n)}](t)$ for each tuple $(a_1, \dots, a_n) \in \mathbf{A}^n$, where $\alpha_{(a_1, \dots, a_n)}(x_i) = a_i$ for $1 \leq i \leq n$. We can define a stable quasi-ordering \succeq on terms given by $t \succeq s$ if and only if $[\alpha](t) \succeq_{\mathbf{A}} [\alpha](s)$, for all $\alpha : \mathcal{X} \rightarrow \mathbf{A}$, where $\succeq_{\mathbf{A}}$ is a quasi-ordering on \mathbf{A} . We can define a stable strict ordering \sqsupset on terms by $t \sqsupset s$ if $[\alpha](t) \succ_{\mathbf{A}} [\alpha](s)$, for all $\alpha : \mathcal{X} \rightarrow \mathbf{A}$, where $\succ_{\mathbf{A}}$ is a strict ordering on \mathbf{A} .

A matrix interpretation for a k -ary symbol f is a linear expression $F_1 x_1 + \dots + F_k x_k + F_0$ where the F_1, \dots, F_k are (square) matrices of $n \times n$ natural numbers and the variables x_1, \dots, x_k (and also the constant term F_0) are n -tuples of natural numbers [10]. An expression like $F\mathbf{x}$, where F is an n -square matrix and \mathbf{x} is an n -tuple of numbers, is interpreted as the usual matrix-vector product, i.e., the i -th component y_i of $\mathbf{y} = F\mathbf{x}$ is $y_i = \sum_{j=1}^n F_{ij} x_j$. Matrices and vectors are compared by using an *entrywise* ordering: for $A, B \in \mathbb{R}^{p \times q}$, we write $A \geq B$ if $A_{ij} \geq B_{ij}$ for all $1 \leq i \leq p$, $1 \leq j \leq q$ [12, Chapter 15]. We also write $A > B$ if $A \geq B$ and $A_{11} > B_{11}$ [10]. Note that this also describes how to compare tuples of numbers. In [1,2], Endrullis et al.'s approach was extended to matrix interpretations with real numbers in matrix entries.

Here, we generalize the notion of matrix interpretation in the following ways:

1. We consider a domain $T_{n,b}(N)$ of *block-based tuples over N* , i.e., n -tuples consisting of $\beta = \frac{n}{b}$ tuples of size $b \in \mathbb{N}_{>0}$ which are *constant* tuples $c\mathbf{1}_b$ for some number $c \in N$. If we take $b = 1$ and $N = \mathbb{N}$ or $N = \mathbb{R}_0$, then we are in the original approaches [10] and [1,2], respectively. Note, however, that given $n \in \mathbb{N}_{>0}$, only divisors b of n can be used to establish blocks within n -tuples of numbers.
2. Matrices F in matrix expressions interpreting symbols $f \in \mathcal{F}$ will be *block matrices* $\begin{pmatrix} F_{11} & \dots & F_{1\beta} \\ \vdots & \ddots & \vdots \\ F_{\beta 1} & \dots & F_{\beta\beta} \end{pmatrix}$ such that $F_{ij} = C_{ij} + S_{ij}$ is a sum of a b -square *constant* matrix $C_{ij} = c_{ij}\mathbf{1}_{b \times b}$, where $c_{ij} \in N$, and a b -square *scalar* matrix $S_{ij} = s_{ij}I_b$, where $s_{ij} \in N$, for all $1 \leq i, j \leq \beta$. This is necessary for soundness of the obtained algebraic interpretation: the product of one of such matrices by a block-based tuple as above produces a block-based tuple as above, i.e., $F\mathbf{v} \in T_{n,b}(N)$ for all $\mathbf{v} \in T_{n,b}(N)$. Furthermore, such matrices are *closed* under addition and matrix product. This is essential to obtain matrices of this kind during the interpretation of the terms, where nested symbols yield products and sums of matrices after interpreting them. Again, if $b = 1$, we are in the usual case for matrix interpretations.
3. Given a matrix valuation ρ , and matrices A, B , we let $A \geq_{\rho} B$ if $\rho(A) \geq_N \rho(B)$, where \geq_N is an ordering over N . Given $\delta > 0$, the following ordering over (real) numbers is used [15]: $x >_{\delta} y$ if $x - y \geq \delta$. If $\delta = 1$ and x, y are natural numbers, we obtain the usual well-founded ordering among natural numbers $>_{\mathbb{N}}$. Now, the following (strict and well-founded) ordering $>_{\rho, \delta}$ (or just $>_{\rho}$ or even $>$ if it is clear from the context) on n -tuples of *nonnegative*

numbers is considered: $\mathbf{x} >_{\rho,\delta} \mathbf{y}$ if $\rho(x) >_{\delta} \rho(y)$. Clearly, $(T_{n,b}(\mathbb{N}), >_{\rho,1})$ and $(T_{n,b}(\mathbb{R}_0), >_{\rho,\delta})$ are well-founded orderings.

The previous orderings do *not* take into account the block structure of matrices or tuples. The following one does: for matrices A, B with blocks A_{ij} and B_{ij} for $1 \leq i, j \leq \beta$, we write $A \geq_{\rho}^b B$ if $A_{ij} \geq_{\rho} B_{ij}$ for all $1 \leq i, j \leq \beta$. Similarly, $A >_{\rho,\delta}^b B$ if $A_{11} >_{\rho,\delta} B_{11}$ and $A_{ij} \geq_{\rho} B_{ij}$ for all $1 \leq i, j \leq \beta$. These definitions are adapted to tuples in the obvious way.

Definition 4 (Block-based matrix interpretation). *Let \mathcal{F} be a signature, $n, b \in \mathbb{N}_{>0}$ be such that b divides n , and $\beta = \frac{n}{b}$. An (n,b) -block-based matrix interpretation is an \mathcal{F} -algebra $\mathcal{A} = (A, \mathcal{F}_{\mathcal{A}})$ such that*

1. $A = T_{n,b}(N)$ for $N = \mathbb{N}$ or $N = \mathbb{R}_0$, and
2. $\mathcal{F}_{\mathcal{A}}$ consists of matrix functions $[f](x_1, \dots, x_k) = F_1 x_1 + \dots + F_k x_k + F_0$ which are closed in A and where, for all $f \in \mathcal{F}$, $1 \leq i \leq k$,
 - (a) F_i is an n -square matrix of $\beta \times \beta$ blocks of b -square matrices $C_{j\ell} + S_{j\ell}$ such that $C_{j\ell}$ is a constant matrix, and $S_{j\ell}$ is a scalar matrix for all $1 \leq j \leq \beta$ and $1 \leq \ell \leq \beta$.
 - (b) $F_0 = (\mathbf{c}_1^T \dots \mathbf{c}_\beta^T)^T$ consists of β b -tuples $\mathbf{c}_j = c_j \mathbf{1}_b$ for some $c_j \in N$.

We make the orderings which are going to be used explicit by adding them to the pair $(A, \mathcal{F}_{\mathcal{A}})$, thus specifying an ordered algebra: $(A, \mathcal{F}_{\mathcal{A}}, \geq_{\rho_n})$, $(A, \mathcal{F}_{\mathcal{A}}, \geq_{\rho_b}^b)$, etc.

6 Solving and Transforming Matrix Constraints

In the following, we consider two kinds of constraint solving problems which are relevant in proofs of termination.

6.1 Testing Universally Quantified Symbolic Constraints

Proofs of termination in term rewriting involve solving *weak* or *strict* symbolic constraints $s \succeq t$ or $s \sqsupset t$ between terms s and t coming from (parts of) the rules of the TRS where the variables in s and t are universally quantified in the corresponding constraint. Here, \succeq and \sqsupset are (quasi-)orderings on terms satisfying appropriate conditions [3,6,8].

Example 9. The following symbolic constraints must be checked in order to guarantee termination of the TRS \mathcal{R} in Example 1:

$$\forall X(f(f(X)) \succeq f(g(f(X)))) \quad (5)$$

$$\forall X(f(g(f(X))) \succeq X) \quad (6)$$

$$\forall X(F(f(X)) \sqsupset F(g(f(X)))) \quad (7)$$

$$\forall X(F(f(X)) \sqsupset F(X)) \quad (8)$$

Here, variables X range on terms in $\mathcal{T}(\mathcal{F}, \mathcal{X})$, \succeq and \sqsupset are intended to be interpreted as a monotonic and stable quasiordering on terms, and a well-founded and stable ordering on terms, respectively.

In the so-called *interpretation method* for checking symbolic constraints as the ones in Example 9, we use appropriate *ordered \mathcal{F} -algebras* to generate the necessary orderings (see Section 5). In our setting, we are interested in investigating the use of matrix algebras. The following result shows how to *decrease* the value of some of the entries in a block-based matrix algebra over the naturals to obtain an *equivalent* block-based matrix algebra which uses bigger matrices.

Theorem 3. *Let \mathcal{F} be a signature and c be a symbolic constraint $\forall x(s \bowtie t)$ for terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Let $\mathcal{A} = (T_{n,b}(\mathbb{N}), \mathcal{F}_{\mathcal{A}}, \geq_{\rho_b}^b)$ be an (n, b) -block-based matrix interpretation over the naturals and N be the maximum of all entries occurring in any matrix or vector in \mathcal{A} . Let $\mathcal{B} = (B, \mathcal{F}_{\mathcal{B}}, \geq_{\rho_{bN}}^{bN})$ be an (Nn, Nb) -block-based matrix interpretation where $B = T_{Nn, Nb}(\mathbb{N})$ and for all $f \in \mathcal{F}$, $[f]_{\mathcal{B}}(x_1, \dots, x_k) = \mu_N(F_1)x_1 + \dots + \mu_N(F_k)x_k + \nu_N(F_0)$ iff $[f]_{\mathcal{A}}(x_1, \dots, x_k) = F_1x_1 + \dots + F_kx_k + F_0$. Then, \mathcal{A} satisfies c if and only if \mathcal{B} satisfies c .*

Example 10. Consider the following TRSs [10, Example 3]:

$$\mathcal{R} : f(a, g(y), z) \rightarrow f(a, y, g(y)) \quad (9)$$

$$f(b, g(y), z) \rightarrow f(a, y, z) \quad (10)$$

$$a \rightarrow b \quad (11)$$

$$\mathcal{S} : f(x, y, z) \rightarrow f(x, y, g(z)) \quad (12)$$

In order to prove termination of \mathcal{R} relative to \mathcal{S} (written $SN(R/S)$), we have to check that $\forall x(l \succeq r)$ holds for all rules $l \rightarrow r \in \mathcal{R} \cup \mathcal{S}$. Endrullis et al. use the following matrix interpretation for that:

$$\begin{aligned} [a] &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} & [f](x, y, z) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} y + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ [b] &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} & [g](x) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

By using Theorem 3, we conclude that the following block-based matrix interpretation with bit matrices of dimension 4 can also do the work.

$$\begin{aligned} [a] &= \begin{pmatrix} \mathbf{1}_2 \\ \mathbf{0}_2 \end{pmatrix} & [f](x, y, z) &= \begin{pmatrix} I_2 & \mathbf{0}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix} x + \begin{pmatrix} I_2 & \mathbf{1}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix} y + \begin{pmatrix} I_2 & \mathbf{0}_{2 \times 2} \\ \mathbf{0}_{2 \times 2} & \mathbf{0}_{2 \times 2} \end{pmatrix} z + \begin{pmatrix} \mathbf{0}_2 \\ \mathbf{0}_2 \end{pmatrix} \\ [b] &= \begin{pmatrix} \mathbf{0}_2 \\ \mathbf{0}_2 \end{pmatrix} & [g](x) &= \begin{pmatrix} I_2 & \mathbf{0}_{2 \times 2} \\ I_2 & I_2 \end{pmatrix} x + \begin{pmatrix} \mathbf{0}_2 \\ \mathbf{1}_2 \end{pmatrix} \end{aligned}$$

6.2 Solving Existentially Quantified Arithmetic Constraints

In many applications, when algebraic interpretations have to be *generated* rather than given by the user, it is usual to work with *parametric* interpretations.

Example 11. For the symbols occurring in \mathcal{R} in Example 1, we can consider the following linear *parametric* interpretation:

$$[f](x) = f_1x + f_0 \quad [g](x) = g_1x + g_0 \quad [F](x) = F_1x + F_0$$

where f_1, g_1 , and F_1 are expected to be n -square matrices for some $n \in \mathbb{N}_{>0}$ (the case $n = 1$ corresponds to a linear polynomial interpretation) and f_0, g_0, F_0 are n -tuples. The satisfaction of the following *arithmetic* constraints³

$$f_1 f_1 \geq f_1 g_1 f_1 \tag{13}$$

$$f_1 f_0 + f_0 \geq f_1 g_1 f_0 + f_1 g_0 + f_0 \tag{14}$$

$$f_1 g_1 f_1 \geq 1 \tag{15}$$

$$f_1 g_1 f_0 + f_1 g_0 + f_0 \geq 0 \tag{16}$$

$$F_1 f_1 \geq F_1 g_1 f_1 \tag{17}$$

$$F_1 f_0 + F_0 > F_1 g_1 f_0 + F_1 g_0 + F_0 \tag{18}$$

$$F_1 f_1 \geq F_1 \tag{19}$$

$$F_1 f_0 + F_0 > 0 \tag{20}$$

is necessary to ensure that \mathcal{R} is terminating.

By a parametric matrix interpretation we mean a matrix interpretation where the entries in matrices are not numbers but rather *parametric coefficients*, i.e., variables for which we have to provide a numeric value, depending on some (existential) constraints.

In general, given a set of variables \mathcal{X} , we consider here symbolic arithmetic constraints of the form $s \bowtie t$ for $\bowtie \in \{\geq, >\}$, where s is of the form $\sum_{i=1}^{m_s} s_i$ for $m_s > 0$ and $s_i = s_{i1} \cdots s_{im_{s,i}}$ with $m_{s,i} > 0$ for all $1 \leq i \leq \sigma_s$ and $s_{ij} \in \mathcal{X}$ (t would have an analogous structure). Furthermore, we assume existential quantification over all variables occurring in s and t . Note in Example 11 that we use *constants* like 0 and 1. They are handled as ‘special’ variables which will receive the intended interpretation.

Consider a *valuation* $\eta : \mathcal{X} \rightarrow N$ for the variables in \mathcal{X} as numbers in N . Here, we consider $N = \mathbb{N} \cup \mathcal{Q}$ for some finite subset of rational numbers $\mathcal{Q} \subseteq \mathbb{Q} - \mathbb{N}$ satisfying some conditions. We are interested in representing numbers $\eta(x)$ as matrices $\mu(\eta(x))$ as discussed above. Note that we *cannot* represent arbitrary rational numbers by using matrices over the naturals (Remark 2). Still, when dealing with finite sets \mathcal{C} of arithmetic restrictions, we can restrict the attention to those rational numbers which are required to check its satisfaction for a given valuation η . This includes not only rational numbers $\eta(x)$ which are assigned to $x \in \mathcal{X}$, but also those which could occur during the evaluation of an arithmetic expression due to products $\eta(x)\eta(y)$ of rational numbers $\eta(x)$ and $\eta(y)$ which have been associated to variables x and y . The idea is that \mathcal{Q} should contain such rational numbers.

Definition 5 (Compatible domain of rational numbers). *Let $\eta : \mathcal{X} \rightarrow \mathbb{N} \cup \mathcal{Q}$ be a valuation for some $\mathcal{Q} \subseteq \mathbb{Q} - \mathbb{N}$ and \mathcal{C} be a set of arithmetic constraints. Given a multiplicative component $s = s_1 \cdots s_m$ of an arithmetic expression in a constraint in \mathcal{C} , let $I_s = \{i_1, \dots, i_k\}$ be the set of indices of variables in s whose*

³ By lack of space, we cannot explain how these constraints are obtained. Full details about this standard procedures can be found elsewhere, see, e.g., [7,10,15,16].

valuation is a rational (and noninteger) number, i.e., for all $i \in I_s$, $\eta(s_i) \in \mathbb{Q}-\mathbb{N}$. We say that \mathcal{Q} is compatible with \mathcal{C} and η if $\prod_{j \in J} \eta(s_j) \in \mathcal{Q}$ for all multiplicative components s in \mathcal{C} and $J \subseteq I_s$ such that $J \neq \emptyset$.

Example 12. The numeric valuation which corresponds to the polynomial interpretation in Example 1 is:

$$\begin{array}{cccc} \eta(1) = 1 & \eta(0) = 0 & \eta(f_1) = 2 & \eta(f_0) = 2 \\ \eta(g_1) = \frac{1}{2} & \eta(g_0) = \frac{1}{2} & \eta(F_1) = 1 & \eta(F_0) = 0 \end{array}$$

The set $\mathcal{Q} = \{\frac{1}{2}\}$ is compatible with this valuation and with the constraints \mathcal{C} in Example 11. Consider $\mathcal{C}' = \mathcal{C} \cup \{f_1 g_1 f_1 \geq f_1 g_1 f_1 g_1\}$. Now, \mathcal{Q} is *not* compatible with \mathcal{C}' and η because we have that $\eta(g_1)\eta(g_1) = \frac{1}{4} \notin \mathcal{Q}$. If we add $\frac{1}{4}$ to \mathcal{Q} , then we get compatibility with \mathcal{C}' .

A valuation η is extended to terms and constraints by $\eta(s \bowtie t) = \eta(s) \bowtie_{\mathbb{Q}} \eta(t)$, $\eta(s_1 + \dots + s_m) = \eta(s_1) + \dots + \eta(s_m)$, and $\eta(x_1 \dots x_m) = \eta(x_1) \dots \eta(x_m)$.

Remark 4. In general, we assume that $+$ is commutative, but (as happens with the matrix product) we do *not* assume commutativity of the product in arithmetic expressions or their valuations.

Now we have to extend μ_n in Definition 3 to deal with rational numbers in \mathcal{Q} .

Remark 5. In contrast to natural numbers, we have no systematic way to associate matrices to rational numbers yet. In Section 4 we have investigated some partial solutions to this problem. In particular, Examples 5, 6, 7, and 8, show that the *dimension* n of the considered matrices and tuples heavily depend on the numbers in \mathcal{Q} . On the other hand, these examples also provide useful encodings for rational numbers which are frequently used in automatic proofs of termination with polynomial or matrix interpretations [5,11,16].

Assume that μ_n has been extended to each $x \in \mathcal{Q}$ in such a way that: for all $x, y \in \mathcal{Q}$ such that $xy \in \mathcal{Q}$, $\rho_n(\mu_n(x)\mu_n(y)) = xy = \rho_n(\mu_n(x))\rho_n(\mu_n(y))$.

In our setting, variables in \mathcal{X} are interpreted not only as matrices but some of them as *vectors*. Assume that $\mathcal{X}_0 \subseteq \mathcal{X}$ must be interpreted in this way and that the constraints in \mathcal{C} are consistent with this, i.e., whenever a variable $x_0 \in \mathcal{X}_0$ occurs in a constraint $c \in \mathcal{C}$ of the form $s \bowtie t$, each multiplicative term in s and t must contain a single variable $y_0 \in \mathcal{X}_0$ which must be at the end of the term.

Example 13. For the constraints in Example 11, we have $\mathcal{X}_0 = \{f_0, F_0, g_0, 0\}$.

The vectorial (or n -tuple) representation of $x \in \mathbb{N} \cup \mathcal{Q}$ by μ_n is $\mu_n(x)\mathbf{1}_n$.

Example 14. The matrices over the naturals which correspond to η and \mathcal{Q} in Example 12 is (with the encoding of $\frac{1}{2}$ in Example 5) are:

$$\begin{array}{lll} \mu(\eta(f_1)) = \mathbf{1}_{2 \times 2} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \mu(\eta(f_0)) = 2 \mathbf{1}_2 = \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} & \mu(\eta(g_1)) = J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \mu(\eta(g_0)) = J_2 \mathbf{1}_2 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} & \mu(\eta(F_1)) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mu(\eta(F_0)) = 0 \mathbf{1}_2 = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} \\ \mu(\eta(1)) = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mu(\eta(0)) = 0 \mathbf{1}_2 = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix} & \end{array}$$

Accordingly, the matrix interpretation over the naturals which corresponds to the polynomial interpretation over the rationals in Example 1 is:

$$[f](x) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad [g](x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad [F](x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

This matrix interpretation induced from the polynomial interpretation over the rationals in Example 1 can also be used to solve the constraints in the example.

Our technique could be used to translate matrix interpretations *over the rationals* into matrix interpretations *over the naturals* by just applying the previous translation to the *entries* of the matrix interpretation over the rationals instead to the coefficients of the polynomial interpretation.

7 Conclusions

We have investigated matrix representations of natural and rational numbers which can be used to simulate the arithmetics of natural and rational numbers, respectively. We have introduced the notion of *numeric matrix representation* (Definition 1) which associates a number to a matrix and viceversa and proved that, by using some specific representations (Definitions 2 and 3), the arithmetic of natural numbers is preserved. Furthermore, we have proved that every matrix interpretation over the naturals has an associated bit matrix interpretation of (usually) bigger size of the same associated value (Corollary 4). We have investigated the representation of rational numbers by using matrices of natural numbers. We have proved that this problem has no general solution but we have found some suitable trade-offs for finite subsets of rational numbers by using nilpotent matrices consisting of Jordan blocks. Then we have introduced the notion of block-based matrix interpretation (Definition 4) which generalizes existing approaches to matrix interpretations. We use it to transform matrix interpretations over the naturals into matrix interpretations over $\{0, 1\}$, and also to transform matrix interpretations over the rationals into matrix interpretations over the naturals.

The question posed in the introduction: *are rational numbers somehow unnecessary when dealing with matrix interpretations?* could not be answered in full generality due to the lack of a general procedure for building matrix representations for arbitrary finite sets of rational numbers. Of course, this is a main topic of further research and we think that we have settled a good starting point in considering the use of combinations of Jordan blocks as in Examples 5, 6, 7, and 8. Nevertheless, our results suggest that the use of matrices over the naturals of *big size* can somehow play the role of rational numbers in interpretations of smaller size, and in particular in linear polynomial interpretations over the rationals. This does *not* mean that implementing polynomial or matrix interpretations over the rationals is not useful anymore and that natural numbers

should be used everywhere. In fact, working with matrix interpretations of big size is computationally expensive. Another interesting consequence of our analysis is the connection between dimension of matrices over the naturals and value of their entries via Proposition 4 and Corollary 4. Roughly speaking, these results can be interpreted by saying that bigger dimensions of matrices permit the use of smaller entries. In practice, most tools put strong numeric bounds to the coefficients or entries of the interpretations. Our results suggest that increasing such bounds could have a similar effect to increasing the size of the matrices. A more precise analysis about the trade-offs in design and efficiency which these considerations could lead to is also subject for future work.

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