Termination of Innermost Context-Sensitive Rewriting Using Dependency Pairs*

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Abstract. Innermost context-sensitive rewriting has been proved useful for modeling computations of programs of algebraic languages like Maude, OBJ, etc. Furthermore, innermost termination of rewriting is often easier to prove than termination. Thus, under appropriate conditions, a useful strategy for proving termination of rewriting is trying to prove termination of innermost rewriting. This phenomenon has also been investigated for context-sensitive rewriting (*CSR*). Up to now, only few transformations have been proposed and used to prove termination of innermost *CSR*. In this paper, we investigate direct methods for proving termination of innermost *CSR*. We adapt the recently introduced context-sensitive dependency pairs approach to innermost *CSR* and show that they can be advantageously used for proving termination of innermost *CSR*. We have implemented them as part of the termination tool MU-TERM.

1 Introduction

The *dependency pairs method* [3] is one of the most powerful techniques for proving termination of Term Rewriting Systems (TRSs [21, 22]). Roughly speaking, given a TRS $\mathcal{R}$, the dependency pairs associated with $\mathcal{R}$ form a new TRS $\mathsf{DP}(\mathcal{R})$ which (together with $\mathcal{R}$) determines the so-called *dependency chains* which characterize termination of $\mathcal{R}$. The dependency pairs can be presented as a *dependency graph*, where the absence of infinite chains can be analyzed by considering the *cycles* in the graph. In [1], the dependency pairs method has been adapted for proving termination of *context-sensitive rewriting* (*CSR* [15, 18]). With *CSR* we can *achieve* a terminating behavior for non-terminating TRSs by pruning (all) infinite rewrite sequences. In *CSR* we only rewrite μ-replacing subterms. Here, μ is a *replacement map*, i.e., a mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \ldots, k\}$, for each k-ary symbol f of the signature $\mathcal{F}$ [15]. We use them to indicate the argument positions on which the rewriting steps are allowed. Then, t_i is a μ-replacing subterm of $f(t_1, \ldots, t_k)$ if $i \in \mu(f)$; every term t (as a

* This work has been partially supported by the EU (FEDER) and the Spanish MEC, under grants TIN 2004-7943-C04-02 and HA 2006-2007, and the Generalitat Valenciana under grant GV06/285. Beatriz Alarcón was partially supported by the Spanish MEC under FPU grant AP2005-3399.

whole) is μ-replacing by definition. For other subterms we proceed inductively in this way. Then, for a given TRS $\mathcal{R}$ and a replacement map μ, we obtain a restriction of rewriting which we call *context-sensitive rewriting*. A pair $(\mathcal{R}, \mu)$ is often called a context-sensitive TRS (CS-TRS). Proving termination of *CSR* is an interesting problem with several applications in the fields of term rewriting and programming languages (see [20] for further motivation). Furthermore, termination of *innermost CSR* (i.e., the variant of *CSR* where only the deepest μ-replacing redexes are contracted) has proved useful for proving termination of programs in programming languages like Maude and OBJ* which permit to control the program execution by means of such context-sensitive annotations [16, 17].

Proving innermost termination of rewriting is often easier than proving termination of rewriting [3] and, for some relevant classes of TRSs, innermost termination of rewriting is even equivalent to termination of rewriting [10, 11]. In [7, 12] it is proved that the equivalence between termination of innermost *CSR* and termination of *CSR* holds in some interesting cases (e.g., for *orthogonal* CS-TRSs).

Example 1. Consider the following orthogonal TRS $\mathcal{R}$ which is a variant of an example in [4]:

```
from(X) -> cons(X,from(s(X)))
sel(0,cons(X,XS)) -> X
sel(s(N),cons(X,XS)) -> sel(N,XS)
minus(X,0) -> X
minus(s(X),s(Y)) -> minus(X,Y)
quot(0,s(Y)) -> 0
quot(s(X),s(Y)) -> s(quot(minus(X,Y),s(Y)))
zWquot(nil,nil) -> nil
zWquot(cons(X,XS),nil) -> nil
zWquot(nil,cons(X:XS)) -> nil
zWquot(cons(X,XS),cons(Y,YS))->cons(quot(X,Y),zWquot(XS,YS))
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together with $\mu(\mathtt{cons}) = \{1\}$ and $\mu(f) = \{1, \ldots, ar(f)\}$ for all other symbols f. According to [6], innermost μ-termination of $\mathcal{R}$ implies its μ-termination as well. We will show how $\mathcal{R}$ can easily be proved innermost μ-terminating (and hence μ-terminating) by using the results in this paper.

In this paper, we extend the context-sensitive dependency pairs approach in [1] for proving termination of innermost *CSR*. Actually, techniques for proving termination of innermost *CSR* have already been investigated [7, 16]. However, these papers only consider *transformational* techniques, where the original CS-TRS $(\mathcal{R}, \mu)$ is transformed into a TRS $\mathcal{R}^{\mu}_{\Theta}$ (where Θ represents the transformation which has been used) whose *innermost* termination implies the innermost termination of *CSR* for $(\mathcal{R}, \mu)$. Up to now, no direct method has been proposed to prove termination of innermost *CSR*. As shown in [2], proofs of termination using context-sensitive dependency pairs (CSDPs) are much more powerful and faster than any other technique for proving termination of *CSR*. Dealing with innermost *CSR*, we have a similar situation.

Example 2. Consider the following TRS $\mathcal{R}$:

```
b -> c(b)
f(c(X),X) -> f(X,X)
```

together with $\mu(\mathtt{f}) = \{1,2\}$ and $\mu(\mathtt{c}) = \varnothing$. This system is *not* μ-terminating:

$$\mathtt{f(\underline{b},b)} \hookrightarrow \underline{\mathtt{f(c(b),b)}} \hookrightarrow \mathtt{f(\underline{b},b)} \hookrightarrow \ldots$$

where $\hookrightarrow$ denotes a context-sensitive rewriting step. However $\mathcal{R}$ is innermost μ-terminating. We can give a very easy automatic proof of this fact because the *innermost* context-sensitive dependency graph has *no cycle*. In contrast, by using available transformations for proving innermost termination of *CSR* (see [8] for a survey), we could not obtain a proof by using tools (like AProVE or TTT) supporting innermost termination proofs (of rewriting).

In the dependency pairs approach, proofs of termination of innermost rewriting are easier than proofs of termination of rewriting because: (1) the estimated innermost dependency graph is more accurate, (2) it is possible to limit the attention on the so-called *usable* rules of the TRS (for a given cycle).

After some preliminaries in Section 2, in Section 3 we prove that termination of innermost *CSR* can be characterized by using an appropriate definition of chain of CSDPs. In Section 4 we show how to prove automatically innermost termination of *CSR* by using the *innermost context-sensitive dependency graph*. Section 5 adapts the notion of usable rules to deal with innermost *CSR*. Section 6 provides a first experimental evaluation of our techniques.

2 Preliminaries

Terms. Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. Given positions p, q, we denote their concatenation as $p.q$. Positions are ordered by the standard prefix ordering $\leq$. If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. We denote the topmost position by Λ. The set of positions of a term t is $\mathcal{P}os(t)$. Positions of non-variable symbols in t are denoted as $\mathcal{P}os_{\mathcal{F}}(t)$ while $\mathcal{P}os_{\mathcal{X}}(t)$ are the positions of variables. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s. We write $t \trianglerighteq s$ if $s = t|_p$ for some $p \in \mathcal{P}os(t)$ and $t \triangleright s$ if $t \trianglerighteq s$ and $t \neq s$. The symbol labelling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with zero or more 'holes' $\Box$ (a fresh constant symbol).

Term rewriting. A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

Context-sensitive rewriting. A mapping $\mu : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [15]. Let $M_{\mathcal{F}}$ be the set of all $\mathcal{F}$-maps (or $M_{\mathcal{R}}$ for the $\mathcal{F}$-maps of a TRS $(\mathcal{F}, R)$). A binary relation R on terms is μ-monotonic if $t\,R\,s$ implies $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k)\,R\,f(t_1, \ldots, t_{i-1}, s, \ldots, t_k)$ for all $f \in \mathcal{F}$, $i \in \mu(f)$, and $t, s, t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. The set of *$\mu$-replacing positions* $\mathcal{P}os^{\mu}(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^{\mu}(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^{\mu}(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^{\mu}(t|_i)$, if $t \notin \mathcal{X}$. The set of *μ-replacing* variables of t is $\mathcal{V}ar^{\mu}(t) = \{x \in \mathcal{V}ar(t) \mid \exists p \in \mathcal{P}os^{\mu}(t), t|_p = x\}$. The μ-replacing subterm relation $\trianglerighteq_{\mu}$ is given by $t \trianglerighteq_{\mu} s$ if there is $p \in \mathcal{P}os^{\mu}(t)$ such that $s = t|_p$. We write $t \triangleright_{\mu} s$ if $t \trianglerighteq_{\mu} s$ and $t \neq s$. In *context-sensitive rewriting* (*CSR* [15]), we (only) contract *μ-replacing* redexes: s μ-rewrites to t, written $s \hookrightarrow_{\mu} t$ (or $s \hookrightarrow_{\mathcal{R},\mu} t$ and even $s \hookrightarrow t$, if $\mathcal{R}$ and μ are clear from the context), if $s \stackrel{p}{\to}_{\mathcal{R}} t$ and $p \in \mathcal{P}os^{\mu}(s)$. A μ-normal form is a term which cannot be μ-rewritten. Let $\mathsf{NF}_{\mu}(\mathcal{R})$ (or just NF_{μ} if no confusion arises) be the set of μ-normal forms of a TRS $\mathcal{R}$. A μ-innermost redex is a redex t whose μ-replacing subterms are μ-normal forms: $t = \sigma(l)$ for some substitution σ and rule $l \to r \in \mathcal{R}$ and for all $p \in \mathcal{P}os^{\mu}(t)$, $t|_p \in \mathsf{NF}_{\mu}$. A term s innermost μ-rewrites to t, written $s \stackrel{i}{\hookrightarrow} t$, if $s \stackrel{p}{\to}_{\mathcal{R}} t$, $p \in \mathcal{P}os^{\mu}(s)$, and $s|_p$ is an μ-innermost redex. A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_{\mu}$ is terminating. A term t is μ-terminating if there is no infinite μ-rewrite sequence $t = t_1 \hookrightarrow_{\mu} t_2 \hookrightarrow_{\mu} \cdots \hookrightarrow_{\mu} t_n \hookrightarrow_{\mu} \cdots$ starting from t. A TRS $\mathcal{R}$ is innermost μ-terminating if $\stackrel{i}{\hookrightarrow}_{\mu}$ is terminating. We write $s \stackrel{i}{\hookrightarrow}^{!}_{\mathcal{R}} t$ if $s \stackrel{i}{\hookrightarrow}^{*}_{\mathcal{R}} t$ and $t \in \mathsf{NF}_{\mu}$.

A pair $(\mathcal{R}, \mu)$ where $\mathcal{R}$ is a TRS and $\mu \in M_{\mathcal{R}}$ is often called a CS-TRS.

Reduction pairs. A reduction pair $(\succeq, \sqsupset)$ consists of a stable and weakly monotonic quasi-ordering $\succeq$, and a stable and well-founded ordering $\sqsupset$ satisfying either $\succeq \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \succeq \subseteq \sqsupset$. Note that *monotonicity is not required* for $\sqsupset$.

3 Termination of innermost *CSR* with dependency pairs

In the following definition, given a term $t = f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^{\sharp}$ to denote the *marked* term $f^{\sharp}(t_1, \ldots, t_k)$, where $f^{\sharp}$ is a new fresh symbol (called *tuple* symbol [3]). Given a signature $\mathcal{F}$, we let $\mathcal{F}^{\sharp}$ be the extension of $\mathcal{F}$ containing all tuple symbols for $\mathcal{F}$: $\mathcal{F}^{\sharp} = \mathcal{F} \cup \{f^{\sharp} \mid f \in \mathcal{F}\}$. Similarly, if $t = f^{\sharp}(t_1, \ldots, t_k)$ is a marked term, we write $t^{\natural}$ to denote the unmarked term $f(t_1, \ldots, t_k)$.

Definition 1 (CS-dependency pairs [1]). *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. We define $\mathsf{DP}(\mathcal{R}, \mu) = \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu) \cup \mathsf{DP}_{\mathcal{X}}(\mathcal{R}, \mu)$ to be the set of* context-sensitive *dependency pairs (CS-DPs) where:*

$$\mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu) = \{l^{\sharp} \to s^{\sharp} \mid l \to r \in R, r \trianglerighteq_{\mu} s, root(s) \in \mathcal{D}, l \ntriangleright_{\mu} s\}$$

and $\mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu) = \{l^\sharp \to x \mid l \to r \in R, x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)\}$. *We extend* $\mu \in M_{\mathcal{F}}$ *into* $\mu^\sharp \in M_{\mathcal{F}^\sharp}$ *by* $\mu^\sharp(f) = \mu(f)$ *if* $f \in \mathcal{F}$, *and* $\mu^\sharp(f^\sharp) = \mu(f)$ *if* $f \in \mathcal{D}$.

Example 3. Consider the CS-TRS $(\mathcal{R},\mu)$ in Example 2. There is only one context-sensitive dependency pair:

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F(c(X),X) -> F(X,X)
```

with $\mu^\sharp(\mathtt{F}) = \{1,2\}$.

In the CS-DP approach, termination of *CSR* is characterized as the absence of infinite chains of CS-DPs [1, Definition 2]. In innermost *CSR*, we only perform reduction steps on *innermost replacing redexes*. Therefore, we have to restrict the definition of chains in order to obtain an appropriate notion corresponding to innermost *CSR*. Regarding innermost reductions, arguments of a redex should be in *normal form* before the redex is contracted and, regarding *CSR*, the redex to be contracted has to be in a *replacing* position.

Definition 2 (Innermost μ-chain). *Given a CS-TRS* $(\mathcal{P},\mu^\sharp)$ *of CS-DPs associated to a CS-TRS* $(\mathcal{R},\mu)$, *an innermost* $(\mathcal{R},\mathcal{P},\mu^\sharp)$*-chain is a sequence of pairs* $u_j \to v_j \in \mathcal{P}$ *such that there is a substitution* σ *such that* $\sigma(u_j) \in \mathsf{NF}_\mu(\mathcal{R})$ *and such that, for all* $j \geq 1$,

1. $\sigma(v_j) \overset{i}{\hookrightarrow}^{!}_{\mathcal{R},\mu^\sharp} \sigma(u_{j+1})$, *if* $u_j \to v_j \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$, *and*
2. *if* $u_j \to v_j = u_j \to x_j \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$, *then there is some* $s_j \in \mathcal{T}(\mathcal{F},\mathcal{X})$ *such that* $\sigma(x_j) \trianglerighteq_\mu s_j$ *and* $s_j^\sharp \overset{i}{\hookrightarrow}^{!}_{\mathcal{R},\mu^\sharp} \sigma(u_{j+1})$.

As usual we assume that different occurrences of dependency pairs do not share any variables (renamings are used if necessary). An innermost $(\mathcal{R},\mathcal{P},\mu^\sharp)$*-chain is* minimal *if for all* $u_j \to v_j \in \mathcal{P}$ *and* $j \geq 1$, $\sigma(v_j)$ *is innermost* μ*-terminating (whenever* $u_j \to v_j \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$*) and* $s_j^\sharp$ *is innermost* μ*-terminating (whenever* $u_j \to v_j \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$*).*

Theorem 1. *A CS-TRS* $(\mathcal{R},\mu)$ *is innermost* μ*-terminating if and only if no infinite minimal innermost* μ*-chain exists.*

Let $\mathcal{M}_{\infty,\mu}$ be a set of minimal non-μ-terminating terms in the following sense [1]: t belongs to $\mathcal{M}_{\infty,\mu}$ if t is non-μ-terminating and every strict μ*-replacing* subterm s of t (i.e., $t \triangleright_\mu s$) is μ-terminating. The proof of this result uses the following result.

Proposition 1. [1, Proposition 1] *Let* $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ *be a TRS and* $\mu \in M_{\mathcal{R}}$. *Then for all* $t \in \mathcal{M}_{\infty,\mu}$, *there exist* $l \to r \in R$, *a substitution* σ *and a term* $u \in \mathcal{M}_{\infty,\mu}$ *such that* $t \overset{>\Lambda}{\hookrightarrow^*} \sigma(l) \overset{\Lambda}{\to} \sigma(r) \trianglerighteq_\mu u$ *and either (1) there is a* μ*-replacing subterm* s *of* r *such that* $u = \sigma(s)$, *or (2) there is* $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$ *such that* $\sigma(x) \trianglerighteq_\mu u$.

Proof. (of Theorem 1) We prove the *if* part by contradiction. We show that for any infinite innermost μ-rewriting sequence we can construct an infinite innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain. Let innermost μ-rewriting below the root be $\stackrel{>i}{\hookrightarrow} = (\stackrel{>\Lambda}{\hookrightarrow} \cap \stackrel{i}{\hookrightarrow})$. If $\mathcal{R}$ is not innermost μ-terminating, then, by Proposition 1, there is a term $t \in \mathcal{M}_{\infty,\mu}$, a rule $l \to r \in R$, a substitution σ, and a term $u \in \mathcal{M}_{\infty,\mu}$ such that $t \stackrel{>i}{\hookrightarrow} \sigma(l) \stackrel{\Lambda}{\to} \sigma(r) \trianglerighteq_\mu u$ (where every immediate replacing subterm of $\sigma(l)$ is a μ-normal form), u is not innermost μ-terminating and either

1. there is a μ-replacing subterm s of r such that $u = \sigma(s)$, or
2. there is $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$ such that $\sigma(x) \trianglerighteq_\mu u$.

In the first case above, we have a dependency pair $l^\sharp \to s^\sharp \in \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$ such that $u = \sigma(s) \in \mathcal{M}_{\infty,\mu}$, i.e., we can start an innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain beginning with $\sigma(l^\sharp) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} \sigma(s^\sharp)$. Note that $\sigma(l^\sharp) \in \mathsf{NF}_\mu(\mathcal{R})$.

In the second case above, since $u \in \mathcal{M}_{\infty,\mu}$, there is a rule $\lambda \to \rho$ such that $u \stackrel{>i}{\hookrightarrow}^! \sigma(\lambda)$ (since we can assume that the variables in this rule do not occur in l, we can use the same –conveniently extended– substitution σ) and $\sigma(\rho)$ contains a subterm in $\mathcal{M}_{\infty,\mu}$. Hence, $u^\sharp \stackrel{i}{\hookrightarrow}^!_{\mathcal{R},\mu^\sharp} \sigma(\lambda^\sharp)$. Furthermore, there is a dependency pair $l^\sharp \to x \in \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$ such that $\sigma(x) \trianglerighteq_\mu u$; thus, according to Definition 2 we can start an $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain beginning with

$$\sigma(l^\sharp) \hookrightarrow_{\mathsf{DP}(\mathcal{R},\mu),\mu^\sharp} u^\sharp$$

and then continuing with a dependency pair $u' \to v'$ such that $u' = \lambda^\sharp$ and $u^\sharp \stackrel{i}{\hookrightarrow}^!_{\mathcal{R},\mu^\sharp} \sigma(u')$. Note that $\sigma(l^\sharp), \sigma(\lambda^\sharp) \in \mathsf{NF}_\mu(\mathcal{R})$.

Thus, in both cases we can start an innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain which could be infinitely extended in a similar way by starting from $u^\sharp$. This contradicts our initial assumption.

On the other hand, in order to show that the criterion is also *necessary* for innermost termination of context-sensitive rewriting we assume that there exists an infinite innermost μ-chain that implies the existence of an infinite innermost μ-rewrite sequence. If there is an infinite innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain, then there is a substitution σ and dependency pairs $u_i \to v_i \in \mathsf{DP}(\mathcal{R},\mu)$ such that considering the first dependency pair $u_1 \to v_1$ in the sequence:

1. If $u_1 \to v_1 \in \mathsf{DP}_\mathcal{F}(\mathcal{R},\mu)$, then $v_1^\natural$ is a μ-replacing subterm of the right-hand-side r_1 of a rule $l_1 \to r_1$ in $\mathcal{R}$. Therefore, $r_1 = C_1[v_1^\natural]_{p_1}$ for some $p_1 \in \mathcal{P}os^\mu(r_1)$ and, since $\sigma(u_1) \in \mathsf{NF}_\mu$, we can perform the innermost μ-rewriting step $t_1 = \sigma(u_1^\natural) \stackrel{i}{\hookrightarrow}_{\mathcal{R},\mu} \sigma(r_1) = \sigma(C_1)[\sigma(v_1^\natural)]_{p_1} = s_1$, where $\sigma(v_1^\natural)^\sharp = \sigma(v_1) \stackrel{i}{\hookrightarrow}^!_{\mathcal{R},\mu^\sharp} \sigma(u_2)$ and $\sigma(u_2)$ also initiates an infinite innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain. Note that $p_1 \in \mathcal{P}os^\mu(s_1)$.
2. If $u_1 \to x \in \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu)$, then there is a rule $l_1 \to r_1$ in $\mathcal{R}$ such that $u_1 = l_1^\sharp$, and $x \in \mathcal{V}ar^\mu(r_1) - \mathcal{V}ar^\mu(l_1)$, i.e., $r_1 = C_1[x]_{q_1}$ for some $q_1 \in \mathcal{P}os^\mu(r_1)$. Furthermore, since there is a subterm s such that $\sigma(x) \trianglerighteq_\mu s$ and $s^\sharp \stackrel{i}{\hookrightarrow}^!_{\mathcal{R},\mu^\sharp} \sigma(u_2)$,

we can write $\sigma(x) = C'_1[s]_{p'_1}$ for some $p'_1 \in \mathcal{P}os^\mu(\sigma(x))$. Therefore, since $\sigma(u_1) = \sigma(l_1)^\sharp \in \mathsf{NF}_\mu$, we can perform the innermost μ-rewriting step $t_1 = \sigma(l_1) \overset{i}{\hookrightarrow}_{\mathcal{R},\mu} \sigma(r_1) = \sigma(C_1)[C'_1[s]_{p'_1}]_{q_1} = s_1$ where $s^\sharp \overset{i}{\hookrightarrow}^!_{\mathcal{R},\mu^\sharp} \sigma(u_2)$ (hence $s \overset{i}{\hookrightarrow}^! u_2^\natural$) and $\sigma(u_2)$ initiates an infinite innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chain. Note that $p_1 = q_1.p'_1 \in \mathcal{P}os^\mu(s_1)$.

Since $\mu^\sharp(f^\sharp) = \mu(f)$, and $p_1 \in \mathcal{P}os^\mu(s_1)$, we have that $s_1 \overset{i}{\hookrightarrow}^!_{\mathcal{R},\mu} t_2[\sigma(u_2)]_{p_1} = t_2$ and $p_1 \in \mathcal{P}os^\mu(t_2)$. Therefore, we can build in that way an infinite μ-rewrite sequence

$$t_1 \overset{i}{\hookrightarrow}_{\mathcal{R},\mu} s_1 \overset{i}{\hookrightarrow}^!_{\mathcal{R},\mu} t_2 \overset{i}{\hookrightarrow}_{\mathcal{R},\mu} \cdots$$

which contradicts the innermost μ-termination of $\mathcal{R}$.

Example 4. Consider again the CS-TRS $\mathcal{R}$ in Example 2. As shown in Example 3, there is only one CS-DP:

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F(c(X),X) -> F(X,X)
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Since $\mu^\sharp(\mathtt{F}) = \{1,2\}$, if a substitution σ satisfies $\sigma(\mathtt{F(c(X),X)}) \in \mathsf{NF}_\mu(\mathcal{R})$, then $\sigma(X) = s$ is in μ-normal form. Assume that the dependency pair is part of an innermost CS-DP-chain. Since there is no way to μ-rewrite $F(s,s)$, there must be $F(s,s) = F(c(t),t)$ for some term t, which means that $s = t$ and $c(t) = s$, i.e., $t = c(t)$ which is not possible. Thus, there is no infinite innermost chain of CS-DPs for $\mathcal{R}$, which is proved innermost terminating by Theorem 1.

Of course, ad-hoc reasonings like in Example 4 do not lead to automation. In the following section we discuss how to prove termination of innermost *CSR* by giving *constraints* on terms that can be solved by using standard methods.

4 Checking innermost μ-termination automatically

The analysis of infinite sequences of dependency pairs can be handled by looking at (the cycles $\mathfrak{C}$ of) the dependency graph associated to the TRS $\mathcal{R}$ [3].

The Innermost Context-Sensitive dependency graph of a TRS $\mathcal{R}$ is the directed graph whose nodes are the CS-dependency pairs; there is an arc from $u \to v$ to $u' \to v'$ if $u \to v$, $u' \to v'$ is an innermost μ-chain.

In [2] we have investigated the structure of context-sensitive sequences in order to improve the CS-dependency graph. A μ-rewrite sequence can proceed in two ways: by means of *visible* parts of the rules that is, μ-replacing subterms in the right-hand sides which are rooted by a defined symbol, or showing up *hidden* non-μ-terminating subterms which are activated by *migrating* variables of a rule $l \to r$, i.e. those variables that are not μ-replacing in l and become μ-replacing in r.

Definition 3 (Hidden symbol [2]). *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_\mathcal{R}$. We say that $f \in \mathcal{F}$ is a* hidden symbol *if there is a rule $l \to r \in R$ where f occurs at a non-μ-replacing position. Let $\mathcal{H}(\mathcal{R},\mu)$ (or just $\mathcal{H}$, if $\mathcal{R}$ and μ are clear from the context) be the set of all hidden symbols in $(\mathcal{R},\mu)$.*

Obviously, as innermost μ-chains are restricted chains (also restricted μ-chains!), the Innermost Context-Sensitive dependency graph (in the following ICSDG or ICS-dependency graph for short) is a subgraph of the dependency graph.

Definition 4 (Innermost CSDG). *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. The innermost context-sensitive dependency graph consists of the set $\mathsf{DP}(\mathcal{R},\mu)$ of context-sensitive dependency pairs and arcs which connect them as follows:*

1. *There is an arc from a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{F}}(\mathcal{R},\mu)$ to a dependency pair $u' \to v' \in \mathsf{DP}(\mathcal{R},\mu)$ if there are subtitutions σ and θ such that $\sigma(v) \overset{i}{\hookrightarrow}^! \theta(u')$ and $\sigma(u)$ and $\theta(u') \in \mathsf{NF}_\mu(\mathcal{R})$.*
2. *There is an arc from a dependency pair $u \to v \in \mathsf{DP}_{\mathcal{X}}(\mathcal{R},\mu)$ to a dependency pair $u' \to v' \in \mathsf{DP}(\mathcal{R},\mu)$ if $root(u')^\natural \in \mathcal{H}(\mathcal{R},\mu)$ and u and $u' \in \mathsf{NF}_\mu(\mathcal{R})$.*

Example 5. Consider the following CS-TRS $\mathcal{R}$ in [6]:

```
f(g(b)) -> f(g(a))        f(a) -> f(a)        a -> b
```

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{g}) = \varnothing$. Then $\mathsf{DP}(\mathcal{R},\mu)$ is:

```
F(g(b)) -> F(g(a))        F(a) -> F(a)        F(a) -> A
```

and $\mu^\sharp(\mathtt{F}) = \{1\}$. The CSDG contains a single cycle $\{$`F(a) -> F(a)`$\}$. However, the ICSDG is empty.

4.1 Approximating the ICSDG

In order to automatically build the Innermost Context-Sensitive Dependency Graph it is necessary to approximate it since for two dependency pairs $u \to v$ and $u' \to v'$ it is undecidable to know if there exist two substitutions σ and θ such that $\sigma(v)$ μ-reduces innermost to $\theta(u')$ and $\sigma(u)$ and $\theta(u')$ are instantiated to μ-normal forms. For this reason, we have to approximate the graph by computing a supergraph containing it in the same way as previous approaches [3, 1]. In the context-sensitive setting, we have adapted functions Cap and Ren to be applied only on *μ-replacing* subterms [1]. On the other hand, in the innermost setting it is not necessary to use Ren since all variables are always instantiated to normal forms and cannot be reduced and $\textsc{Cap}(v)$ substitutes every subterm with a defined root symbol by fresh variables only if the term is not equal to subterms of u. To approximate the ICS-dependency graph, however, we have to combine both of them: we use $\textsc{Cap}^\mu_u(v)$ to replace all μ-replacing subterm rooted with a defined symbol whenever the term was not equal to a μ-replacing subterm of the left-hand side of the dependency pair u. We use $\textsc{Ren}^\mu_u(v)$ to replace by fresh variables those ones that are replacing in v but not in u since they are not μ-normalized. Given a term u, we let $\textsc{Cap}^\mu_u$ be given as follows: let D be a set of defined symbols (in our context, $D = \mathcal{D} \cup \mathcal{D}^\sharp$):

$$\textsc{Cap}^\mu_u(x) = x \quad \text{if } x \text{ is a variable}$$
$$\textsc{Cap}^\mu_u(f(t_1,\ldots,t_k)) = \begin{cases} y & \text{if } f \in D \\ f([t_1]^f_1,\ldots,[t_k]^f_k) & \text{otherwise} \end{cases}$$

where y is a new, fresh variable which has not yet been used and given a term s, $[s]^f_i = \textsc{Cap}^\mu_u(s)$ if $i \in \mu(f)$ and s is not equal to a μ-replacing subterm of u and $[s]^f_i = s$ otherwise. Given a term u, we let $\textsc{Ren}^\mu_u$ be given by: $\textsc{Ren}^\mu_u(x) = y$ if x is a variable and $\textsc{Ren}^\mu_u(f(t_1,\ldots,t_k)) = f([t_1]^f_1,\ldots,[t_k]^f_k)$ for evey k-ary symbol f, where given a term $s \in \mathcal{T}^\sharp(\mathcal{F},\mathcal{X})$, $[s]^f_i = \textsc{Ren}^\mu_u(s)$ if $i \in \mu(f)$ and the variable is not μ-replacing in u and $[s]^f_i = s$ otherwise.

We have an arc from $u \to v$ to $u' \to v'$ in the ICS-dependency graph if $\textsc{Ren}^\mu_u(\textsc{Cap}^\mu_u(v))$ and u' are unifiable by some mgu σ such that $\sigma(u), \sigma(u') \in \mathsf{NF}_\mu(\mathcal{R})$; following [3], we say that v and u' are *innermost μ-connectable*. The following result whose proof is similar to that of [3, Theorem 39] formalizes the correctness of this approach (we only need to take into account the replacement restrictions indicated by the replacement map μ).

Proposition 2. *Let $(\mathcal{R},\mu)$ be a CS-TRS. If there is an arc from $u \to v$ to $u' \to v'$ in the ICS-dependency graph, then v and u' are innermost μ-connectable.*

Example 6. (Continuing Example 2) Since $\textsc{Ren}^{\mu^\sharp}_u(\textsc{Cap}^{\mu^\sharp}_u(\texttt{F(X,X)})) = \texttt{F(X,X)}$ and `F(c(Y),Y)` do not unify we conclude (and this can easily be implemented) that the ICS-dependency graph for the CS-TRS $(\mathcal{R},\mu)$ in Example 2 contains no cycles.

We know how to approximate the ICS-dependency graph by means of the functions $\textsc{Cap}^\mu_u$ and $\textsc{Ren}^\mu_u$. The next step is checking the innermost μ-termination with the ICSDG automatically.

4.2 Proofs of termination of innermost *CSR* using the ICSDG

The absence of infinite innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R},\mu), \mu^\sharp)$-chains is checked in the ICSDG by finding (possibly different) μ-reduction pairs $(\gtrsim_\mathfrak{C}, \sqsupset_\mathfrak{C})$ for each cycle $\mathfrak{C}$. Here, a μ-reduction pair is a pair $(\gtrsim, \sqsupset)$ where $\gtrsim$ is a stable and μ-monotonic quasi-ordering which is compatible with the well-founded and stable ordering $\sqsupset$, i.e., satisfying either $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$.

Theorem 2 (Use of the ICSDG). *Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$. Then, $\mathcal{R}$ is innermost μ-terminating iff for each cycle $\mathfrak{C}$ in the innermost context-sensitive dependency graph there is a μ-reduction pair $(\gtrsim_\mathfrak{C}, \sqsupset_\mathfrak{C})$ such that $\mathcal{R} \subseteq \gtrsim_\mathfrak{C}$, $\mathfrak{C} \subseteq \gtrsim_\mathfrak{C} \cup \sqsupset_\mathfrak{C}$, and*

1. *If $\mathfrak{C} \cap \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu) = \varnothing$, then $\mathfrak{C} \cap \sqsupset_\mathfrak{C} \neq \varnothing$*
2. *If $\mathfrak{C} \cap \mathsf{DP}_\mathcal{X}(\mathcal{R},\mu) \neq \varnothing$, then $\trianglerighteq_\mu \subseteq \gtrsim_\mathfrak{C}$, and*
   (a) $\mathfrak{C} \cap \sqsupset_\mathfrak{C} \neq \varnothing$ and $f(x_1,\ldots,x_k) \gtrsim_\mathfrak{C} f^\sharp(x_1,\ldots,x_k)$ for all $f^\sharp$ in $\mathfrak{C}$, or
   (b) $f(x_1,\ldots,x_k) \sqsupset_\mathfrak{C} f^\sharp(x_1,\ldots,x_k)$ for all $f^\sharp$ in $\mathfrak{C}$.

The proof is similar to that of [1, Theorem 4]. The practical use of Theorem 2 concerns the so-called *strongly connected components* (SCCs) of the dependency graph, rather than the cycles themselves (which are exponentially many) [13, 14].

Example 7. There are many examples that are easily solved when trying to build the ICS-dependency graph since they do not contain cycles. This is the case for Example 2 and Example 5.

The use of *argument filterings*, which is standard in the current formulations of the dependency pairs method, also adapts without changes to this setting.

Also the *subterm criterion* [13], can be used to ignore certain cycles of the dependency graph. In [1], we have adapted it to *CSR*.

5 Usable CS-rules

An interesting feature in the treatment of innermost termination problems using the dependency pairs approach is that, since the variables in the right-hand side of the dependency pairs are in normal form, the rules which can be used to connect contiguous dependency pairs are usually a proper subset of the rules in the TRS. This leads to the notion of *usable rules* [3, Definition 32] which simplifies the proofs of innermost termination of rewriting. We adapt this notion to the context-sensitive setting.

Definition 5 (Basic usable CS-rules). *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. For any symbol f let $Rules(\mathcal{R}, f)$ be the set of rules defining f and such that the left-hand side l has no redex as proper μ-replacing subterm. For any term t the set of* basic *usable rules $\mathbf{U}_0(\mathcal{R}, t)$ is as follows:*

$$\begin{aligned}\mathbf{U}_0(\mathcal{R}, x) &= \varnothing \\ \mathbf{U}_0(\mathcal{R}, f(t_1, \ldots, t_n)) &= Rules(\mathcal{R}, f) \cup \bigcup_{i \in \mu(f)} \mathbf{U}_0(\mathcal{R}', t_i) \cup \bigcup_{l \to r \in Rules(\mathcal{R}, f)} \mathbf{U}_0(\mathcal{R}', r)\end{aligned}$$

where $\mathcal{R}' = \mathcal{R} - Rules(\mathcal{R}, f)$. If $\mathfrak{C} \subseteq \mathsf{DP}(\mathcal{R}, \mu)$, then $\mathbf{U}_0(\mathcal{R}, \mathfrak{C}) = \bigcup_{l \to r \in \mathfrak{C}} \mathbf{U}_0(\mathcal{R}, r)$.

Interestingly, although our definition is a straightforward extension of the classical one (which just takes into account that μ-rewritings are possible only on μ-replacing subterms), some subtleties arise due to the presence of *non-conservative* rules. Here, a rule $l \to r$ of a TRS $\mathcal{R}$ is μ-conservative if $\mathcal{V}ar^{\mu}(r) \subseteq \mathcal{V}ar^{\mu}(l)$, i.e., it does not contain migrating variables; $\mathcal{R}$ is μ-conservative if all its rules are (see [20]).

Definition 6 (Conservative CSDPs). *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. The set of* conservative *CSDPs $\mathsf{DP}_{\mathsf{Co}}(\mathcal{R}, \mu)$ is $\mathsf{DP}_{\mathsf{Co}}(\mathcal{R}, \mu) = \{u \to v \in \mathsf{DP}(\mathcal{R}, \mu) \mid \mathcal{V}ar^{\mu^\sharp}(v) \subseteq \mathcal{V}ar^{\mu^\sharp}(u)\}$.*

Note that $\mathsf{DP}_{\mathsf{Co}}(\mathcal{R}, \mu) \subseteq \mathsf{DP}_{\mathcal{F}}(\mathcal{R}, \mu)$. Basic usable rules in Definition 5 can be applied to cycles $\mathfrak{C}$ consisting of *conservative* CS-dependency pairs provided that $\mathbf{U}_0(\mathcal{R}, \mathfrak{C})$ is also conservative. This is proved in Theorem 3 below. First, we need some auxiliary results.

Proposition 3. *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution such that $s = \sigma(t)$ and $\forall x \in \mathcal{V}ar^{\mu}(t)$, $\sigma(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$. If $s \overset{i}{\hookrightarrow} s'$ by applying a rule $l \to r \in \mathcal{R}$, then there is a substitution σ' such that $s' = \sigma'(t')$ for $t' = t[r]_p$ and $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$.*

Proof. Let $p \in \mathcal{P}os^\mu(s)$ be the position of an innermost redex $s|_p = \theta(l)$ for some substitution θ. Since $s = \sigma(t)$ and for all replacing variables in t, we have $\sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$, it follows that p is a non-variable (replacing) position of t. Therefore, $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$. Since $s = \sigma(t)$, we have that $s' = \sigma(t)[\theta(r)]_p$ and since $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$, by defining $\sigma'(x) = \sigma(x)$ for all $x \in Var(t)$ and $\sigma(x) = \theta(x)$ for all $x \in \mathcal{V}ar(r)$ (as usual, we assume $\mathcal{V}ar(t) \cap \mathcal{V}ar(r) = \varnothing$), we have $s' = \sigma'(t[r]_p)$.

Proposition 4. *Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution such that $s = \sigma(t)$ and $\forall x \in \mathcal{V}ar^\mu(t)$, $\sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$. If $s \stackrel{i}{\hookrightarrow} s'$ by applying a conservative rule $l \to r \in \mathcal{R}$, then there is a substitution σ' such that $s' = \sigma'(t')$ for $t' = t[r]_p$, $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$ and $\forall x \in \mathcal{V}ar^\mu(t')$, $\sigma'(x) \in \mathsf{NF}_\mu(\mathcal{R})$.*

Proof. By Proposition 3, we know that σ', as in Proposition 3, satisfies $s' = \sigma'(t')$ for θ as in Proposition 3 and some $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$. Since $s|_p$ is an innermost μ-replacing redex, we have that $\forall y \in \mathcal{V}ar^\mu(l)$, $\theta(y) \in \mathsf{NF}_\mu(\mathcal{R})$. Since the rule $l \to r$ is conservative, $\mathcal{V}ar^\mu(r) \subseteq \mathcal{V}ar^\mu(l)$, hence $\forall z \in \mathcal{V}ar^\mu(r)$, $\sigma'(z) \in \mathsf{NF}_\mu(\mathcal{R})$. Since $\mathcal{V}ar^\mu(t[r]_p) \subseteq \mathcal{V}ar^\mu(t) \cup \mathcal{V}ar^\mu(r)$, we have that $\forall x \in \mathcal{V}ar^\mu(t')$, $\sigma'(x) \in \mathsf{NF}_\mu(\mathcal{R})$.

Proposition 5. *Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution such that $s = \sigma(t)$ and $\forall x \in \mathcal{V}ar^\mu(t)$, $\sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$. If $\mathbf{U}_0(\mathcal{R}, t)$ is conservative and $s \stackrel{i}{\hookrightarrow}^*_\mathcal{R} u$ then $s \stackrel{i}{\hookrightarrow}^*_{\mathbf{U}_0(\mathcal{R},t)} u$.*

Proof. By induction on the length of the sequence $s \stackrel{i}{\hookrightarrow}^*_\mathcal{R} u$. If $s = \sigma(t) = u$, it is trivial. Otherwise, if $s \stackrel{i}{\hookrightarrow}_\mathcal{R} s' \stackrel{i}{\hookrightarrow}^*_\mathcal{R} u$, we first prove that the result also holds in $s \stackrel{i}{\hookrightarrow}_\mathcal{R} s'$. By Proposition 3, $s = \sigma(t)$, and $s' = \sigma'(t')$ for $t' = t[r]_p$ is such that $s|_p = \theta(l)$ and $s'|_p = \theta(r)$ for some $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$. Thus, $root(l) = root(t|_p)$ and by Definition 5, we can conclude that $l \to r \in \mathbf{U}_0(\mathcal{R}, t)$. By hypothesis, $\mathbf{U}_0(\mathcal{R}, t)$ is conservative. Thus, $l \to r$ is conservative and by Proposition 4, $s' = \sigma'(t')$ and $\forall x \in \mathcal{V}ar^\mu(t')$, $\sigma'(x) \in \mathsf{NF}_\mu(\mathcal{R})$. Since $t' = t[r]_p$ and $root(t|_p) = root(l)$, we have that $\mathbf{U}_0(\mathcal{R}, t') \subseteq \mathbf{U}_0(\mathcal{R}, t)$ and (since $\mathbf{U}_0(\mathcal{R}, t)$ is conservative) $\mathbf{U}_0(\mathcal{R}, t')$ is conservative as well. By the induction hypothesis we know that $s' \stackrel{i}{\hookrightarrow}^*_{\mathbf{U}_0(\mathcal{R},t')} u$. Thus we have $s \stackrel{i}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},t)} s' \stackrel{i}{\hookrightarrow}^*_{\mathbf{U}_0(\mathcal{R},t)} u$ as desired.

Theorem 3. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_\mathcal{F}$, and $\mathfrak{C} \subseteq \mathsf{DP}_{\mathsf{Co}}(\mathcal{R}, \mu)$. If there is a $\mu^\sharp$-reduction pair $(\gtrsim, \sqsupset)$ such that, $\mathbf{U}_0(\mathcal{R}, \mathfrak{C})$ is conservative, $\mathbf{U}_0(\mathcal{R}, \mathfrak{C}) \subseteq \gtrsim$, $\mathfrak{C} \subseteq \gtrsim \cup \sqsupset$, and $\mathfrak{C} \cap \sqsupset \neq \varnothing$, then there is no minimal innermost $(\mathcal{R}, \mathfrak{C}, \mu^\sharp)$-chain.*

Proof. We proceed by contradiction. If $\mathcal{R}$ is not innermost μ-terminating, then by Theorem 1 there is an infinite innermost $(\mathcal{R}, \mathsf{DP}(\mathcal{R}, \mu), \mu^\sharp)$-chain:

$$\sigma(u_1) \hookrightarrow_{\mathsf{DP}(\mathfrak{C},\mu),\mu^\sharp} \sigma(v_1) \stackrel{i}{\hookrightarrow}^!_\mathcal{R} \sigma(u_2) \hookrightarrow_{\mathsf{DP}(\mathfrak{C},\mu),\mu^\sharp} \sigma(v_2) \stackrel{i}{\hookrightarrow}^!_\mathcal{R} \sigma(u_3) \hookrightarrow_{\mathsf{DP}(\mathfrak{C},\mu),\mu^\sharp} \cdots$$

for a substitution σ and $u_i \to v_i \in \mathsf{DP}_\mathcal{F}(\mathfrak{C}, \mu)$ for $i \geq 1$. Since $u_i \to v_i \in \mathsf{DP}_{\mathsf{Co}}(\mathcal{R}, \mu)$, and $\sigma(u_i) \in \mathsf{NF}_\mu(\mathcal{R})$, this implies that $\forall x \in \mathcal{V}ar^\mu(v_i)$, $\sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$ and by Proposition 5 the sequence can be seen as:

$$\sigma(u_1) \hookrightarrow_{\mathsf{DP}(\mathfrak{C},\mu),\mu^\sharp} \sigma(v_1) \stackrel{i}{\hookrightarrow}^!_{(\mathbf{U}_0(\mathcal{R},\mathfrak{C}),\mu^\sharp)} \sigma(u_2) \hookrightarrow_{\mathsf{DP}(\mathfrak{C},\mu),\mu^\sharp} \sigma(v_2) \stackrel{i}{\hookrightarrow}^!_{(\mathbf{U}_0(\mathcal{R},\mathfrak{C}),\mu^\sharp)} \sigma(u_3) \hookrightarrow \cdots$$

By stability of $\sqsupset$, we have $\sigma(u_i) \sqsupset \sigma(v_i)$ and by stability, μ-monotonicity and transitivity of $\gtrsim$ we have that $\sigma(v_i) \gtrsim \sigma(u_{i+1})$. By using the compatibility conditions of the μ-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$.

Unfortunately, dealing with non-conservative CSDPs, considering the basic usable CS-rules does *not* ensure a correct approach.

Example 8. Consider again the TRS $\mathcal{R}$:

```
b -> c(b)
f(c(X),X)-> f(X,X)
```

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{c}) = \varnothing$. There are *two* non-conservative CS-DPs (note that $\mu^\sharp(\mathtt{F}) = \mu(\mathtt{f}) = \{1\}$):

```
F(c(X),X) -> F(X,X)
F(c(X),X) -> X
```

and only one cycle in the ICSDG:

```
F(c(X),X) -> F(X,X)
```

Note that $\mathbf{U}_0(\mathcal{R}, \mathtt{F(X,X)}) = \varnothing$. Since this CS-DP is strictly compatible with, e.g., an LPO, we would conclude the innermost μ-termination of $\mathcal{R}$. However, this system is *not* innermost μ-terminating:

$$\mathtt{f(\underline{b},b)} \overset{i}{\hookrightarrow} \underline{\mathtt{f(c(b),b)}} \overset{i}{\hookrightarrow} \mathtt{f(\underline{b},b)} \overset{i}{\hookrightarrow} \cdots$$

The problem is that we have to take into account the special status of variables in the right-hand side of a non-conservative CS-DP. Instances of such variables are *not* guaranteed to be μ-normal forms. For this reason, when a cycle contains at least one non-conservative CS-DP, we have to consider the whole set of rules of the system.

Furthermore, conservativeness of $\mathbf{U}_0(\mathcal{R}, \mathfrak{C})$ cannot be dropped either since we could infer an incorrect result as shown by the following example.

Example 9. Consider the TRS $\mathcal{R}$:

```
b -> c(b)
f(c(X),X) -> f(g(X),X)
g(X) -> X
```

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{g}) = \mu(\mathtt{c}) = \varnothing$. There is only one conservative cycle: $\{\mathtt{F(c(X),X) \text{ -> } F(g(X),X)}\}$ having only one usable (but non-conservative!) rule `g(X) -> X`. This is compatible with the μ-reduction pair induced by the following polynomial interpretation:

$$[\mathtt{f}](x,y) = 0 \qquad [\mathtt{c}](x) = x + 1 \qquad [\mathtt{g}](x) = x \qquad [\mathtt{F}](x,y) = x$$

However the system is not innermost μ-terminating:

$$\underline{\mathtt{f(c(b),b)}} \overset{i}{\hookrightarrow} \mathtt{f(\underline{g(b)},b)} \overset{i}{\hookrightarrow} \mathtt{f(\underline{b},b)} \overset{i}{\hookrightarrow} \underline{\mathtt{f(c(b),b)}} \overset{i}{\hookrightarrow} \cdots$$

Nevertheless, Theorem 3 is useful to improve the proofs of termination of innermost *CSR* as the following example shows.

Example 10. Consider again the TRS $\mathcal{R}$ in example 1. The system contains three cycles in the ICSDG:

```
{ SEL(s(N),cons(X,XS)) -> SEL(N,XS) }
{ MINUS(s(X),s(Y)) -> MINUS(X,Y) }
{ QUOT(s(X),s(Y)) -> QUOT(minus(X,Y),s(Y)) }
```

The first two cycles can be solved by using the subterm criterion. However, without the notion of usable rules, the last one is difficult to solve. The cycle is conservative and the obtained usable rules are also conservative: `minus(X,0) -> X` and `minus(s(X),s(Y)) -> minus(X,Y)`. According to Theorem 3, the cycle can be easily solved by using a polynomial interpretation:

$$[\mathtt{minus}](x,y) = x \qquad [\mathtt{0}] = 0$$
$$[\mathtt{s}](x) = x+1 \qquad [\mathtt{QUOT}](x,y) = x$$

6 Experiments

We have implemented the techniques described in the previous sections as part of the tool MU-TERM [19]. In order to evaluate the techniques which are reported in this paper we have made some benchmarks. We have considered the examples in the Termination Problem Data Base (TPDB, version 3.2) available through the URL:

`http://www.lri.fr/~marche/tpdb/`

Although there is no special TPDB category for innermost termination of *CSR* (yet) we have used the TRS/CSR directory in order to test our techniques for proving termination of innermost *CSR* (Theorems 2, 3). It contains 90 examples of CS-TRSs. We are able to give an automatic proof of innermost μ-termination for 62 examples. In order to evaluate our direct techniques in comparison with the transformational approach of [7, 8, 16], where termination of innermost *CSR* for a CS-TRS ($\mathcal{R}$, μ) is proved by proving innermost termination of a transformed TRS $\mathcal{R}^{\mu}_{\Theta}$, where Θ specifies a particular transformation (see [6, 7] for a survey on this topic), we have transformed the set of examples by using the transformations that are correct for proving innermost termination of *CSR*: Giesl and Middeldorp's correct transformations for proving termination of innermost *CSR*, see [7], although we use the 'authors-based' notation introduced in [20]: GM and C for transformations 1 and 2 for proving termination of *CSR* introduced in [8], and iGM for the specific transformation for proving termination of innermost *CSR* introduced in [7]. Then we have proved innermost termination of the set of examples with AProVE [9], which is able to prove innermost termination of standard rewriting. In fact, AProVE is currently the most powerful tool for proving termination and innermost termination of TRSs but as we have said, MU-TERM is nowadays the only termination tool that proves innermost termination of *CSR*. The results are summarized in Table 1. Further details can be found here:

`http://www.dsic.upv.es/~balarcon/FroCoS07/benchmarks`

Indirectly, we have also made the first benchmarks to evaluate the existing correct transformations for proving innermost termination of *CSR* (see Table 1) showing that, quite surprisingly, the iGM transformation (which is in principle the more suitable one for proving innermost termination of *CSR*) obtains worse results than GM.

	iCSDPs	Transformations
YES score	62	44
YES average time	0.13 sec.	5 sec.

	C	GM	iGM
YES score	24	41	30

Table 1. Comparing techniques for proving termination of innermost *CSR*

7 Conclusions and future work

In this paper, we have extended the context-sensitive dependency pairs approach in [1] for proving termination of innermost *CSR*. We have introduced the notion of an innermost μ-chain (Definition 2) and proved that it can be used to characterize innermost μ-termination (Theorem 1). We have also shown how to automatically prove innermost μ-termination by means of the ICS-dependency graph (Definition 4, Theorem 2). We have formulated the notion of basic usable rules showing how to use them in proofs of innermost termination of *CSR* (Definition 5, Theorem 3). We have implemented these techniques in MU-TERM and have made some benchmarks.

Up to now, no direct method has been proposed to prove termination of innermost *CSR*. So this is the first proposal of a direct method for proving termination of innermost *CSR*. We have extended Arts and Giesl's approach to prove innermost termination of TRSs to *CSR* (thus also extending [1, 2]). The main issue which is left open is a general notion of *usable rules*, which can be used with non-conservative CSDPs. As in the standard case, this would probably help us to achieve better results. Even without them, though, our benchmarks show that the use of CSDPs dramatically improves the performance of existing (transformational) methods for proving termination of innermost *CSR*.

Acknoledgements. We thank the anonymous referees for many useful remarks.

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