Strong and NV-sequentiality of constructor systems*

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Abstract

Constructor Systems (CSs) are an important subclass of Term Rewriting Systems (TRSs) which can be used as an abstract model of some programming languages. While normalizing strategies are always desirable for achieving a good computational behavior of programs, when dealing with lazy languages *infinitary* normalizing strategies should be considered instead since (finite approximations of) infinite data structures can be returned as the result of computations. We have shown that NV-sequential TRSs (hence strongly sequential TRSs, a subclass of them) provide an appropriate basis for the effective definition of normalizing and infinitary normalizing strategies. In this paper, we show that strongly sequential and NV-sequential CSs coincide. Since the implementation of NV-sequential TRSs has been underexplored in comparison to strongly sequential TRSs, this coincidence suggests that, in programming languages, it is a good option to implement NV-sequentiality as strong sequentiality.

Keywords: programming languages, sequentiality, strategies, term rewriting.

1 Introduction

Constructor Systems (CSs) are an important subclass of TRSs which model more sophisticated programming languages [BN98, PE93]. The left-hand side of a rewrite rules of a CS consists of a *defined* function symbol that applies on *patterns* built only from *constructor* symbols and variables (see Example 1 below). This provides a clear distinction between operations and data constructor symbols, where functions are described as 'data transformers'. This is the usual approach in most functional and algebraic languages. While the definition of normalizing strategies (i.e., concrete reduction sequences that obtain a normal form whenever it exists) is always desirable for achieving a good computational behavior of programs, in the context of lazy languages *infinitary normalizing* strategies should be considered instead, since infinite data structures (and their finite approximations) can be returned as the result of certain computations.

Example 1 *Consider the following CS $\mathcal{R}$ to generate prime numbers [KdV03] (we use* 2, 3, ... *instead of* s(s(0)), s(s(s(0))), *etc.).*

```
primes                 → sieve(nats(2))
nats(n)                → n:nats(s(n))
sieve(0:y)             → sieve(y)
sieve(s(n):y)          → s(n):sieve(filter(y,n,n))
filter(x:y,0,m)        → 0:filter(y,m,m)
filter(x:y,s(n),m)     → x:filter(y,n,m)
```

The following (meaningless) reduction sequence:

```
primes →* sieve(2:nats(3))
       →* sieve(2:3:nats(4))
       →* sieve(2:3:4:nats(5))
       →  ···
```

always contracts needed redexes[1] *[HL91] (every redex is trivially needed since* primes *has no normal form). However, the following sequence:*

```
primes →* sieve(2:nats(3))
       →  2:sieve(filter(nats(3),1,1))
       →* 2:3:sieve(···)
       →  ···
```

*actually 'obtains' (*converges to*) the infinite normal form* 2:3:5:7:11:··· *(the list of prime numbers).*

Example 1 shows that the usual approach to define normalizing strategies based on reducing *needed redexes* does not work in infinitary normalization. Fortunately, it is still possible to effectively define an appropriate strategy for $\mathcal{R}$ in Example 1 that can be used to generate a rewrite sequence that approximates the desired infinite normal form of primes: $\mathcal{R}$ is easily proved

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[1] A redex in a term is *needed* if the redex (itself or one of its descendants) is reduced in each rewriting sequence leading to a normal form.

strongly sequential [HL91, KM91] (see Section 4 below) and, hence, it admits a reduction strategy that reduces the 'appropriate' redexes in such a way that the desired convergent infinitary sequence is obtained [Luc98].

In order to effectively define normalizing strategies based on reducing needed redexes, decidable approximations to the (undecidable) Huet and Lévy's notion of neededness have been extensively explored [Com00, DM97, HL91, Jac96, JS94, KM91, Nag99, NST95, Oya93, Toy92]. We have investigated the use of these approximations to provide an adequate basis for the effective definition of normalizing and infinitary normalizing strategies. We have demonstrated that NV-sequentiality [Oya93] is the most general approximation (among them) which is still appropriate for infinitary normalization. Strong sequentiality [HL91] is a particular case of NV-sequentiality.

In this paper, we show that strong and NV-sequential CSs coincide. First we consider *orthogonal* CSs for which the equivalence easily follows from standard results in [HL91, KM91, Oya93]. Then, we consider overlapping CSs. As strong sequentiality was initially defined for orthogonal TRSs, we show that (surprisingly) there are two different (but related) notions of strong sequentiality of left-linear (possibly overlapping) TRSs in the literature. The first one is Toyama's extension of strong sequentiality from orthogonal to left-linear, possibly overlapping TRSs which was developed in [Toy92] as the natural extension of the classical definition of sequentiality [HL91, Jac96] (see also [Com95, Com00]). The second one is Jouannaud and Sadfi's definition [JS94] which considers a less restrictive approximation. As we show in this paper, the second approach (strictly) includes the first one. We also prove that Jouannaud and Sadfi's notion of strong sequentiality of left-linear CSs is equivalent to NV-sequentiality of left-linear CSs. The equivalence does not hold for strong and *nv*-indices, i.e., the kind of redexes which we need to use in each class of TRSs to achieve the intended computational behavior. However, we show that, concerning the definition of normalizing or infinitary normalizing reduction strategies, strong index reduction in strongly sequential CSs and *nv*-index reduction in NV-sequential CSs are essentially equivalent. Therefore, we claim that concerning e.g., the implementation or efficiency of NV-sequentiality in CSs no special effort is necessary since currently developed implementations based on (possibly restricted kinds of) strong sequentiality (see, e.g., [Dur94, HLM98, Str89]) are appropiate. Our results also suggest that strongly sequential TRSs are the most important class of TRSs which provide complete implementations of rewriting-based programming languages (using CSs in most cases).

After some preliminary definitions, in Section 3 we recall the notion of sequentiality and their approximations. Sections 4 and 5 relate strong and NV-sequentiality of (respectively) orthogonal and left-linear CSs. Section 6 addresses the problem of defining index reduction strategies for strong and NV-sequential CSs. Section 7 concludes.

2 Preliminaries

Throughout the paper $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a function $ar : \mathcal{F} \to \mathbb{N}$. We denote the set of terms by $\mathcal{T}(\mathcal{F}, \mathcal{X})$. $\mathcal{V}ar(t)$ is the set of variables in t. Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers which are used to address subterms of t. The empty chain is denoted by Λ. We use the standard prefix ordering on positions: $p \leq q$ iff $\exists q'$ such that $q = p.q'$ (and we denote q' by p/q); $p \parallel q$ means $p \not\leq q$ and $q \not\leq p$. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced with s. We denote the set of positions of a term t by $\mathcal{P}os(t)$. A term is said to be linear if it has no multiple occurrences of a single variable. A substitution is a mapping $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$ which homomorphically extends to a mapping $\sigma : \mathcal{T}(\mathcal{F}, \mathcal{X}) \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. We write $\sigma \leq \sigma'$ if there is θ such that $\sigma' = \theta \circ \sigma$. A unifier of two terms t_1, t_2 is a substitution σ with $\sigma(t_1) = \sigma(t_2)$. A most general unifier (*mgu*) of t_1, t_2 is a unifier σ with $\sigma \leq \sigma'$ for all other unifiers σ' of t_1, t_2.

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ where R is a set of rewrite rules. $L(\mathcal{R})$ denotes the *lhs*'s of $\mathcal{R}$. An instance $\sigma(l)$ of a *lhs* l of a rule is a redex. The set of redex positions in t is $\mathcal{P}os_{\mathcal{R}}(t) = \{p \in \mathcal{P}os(t) \mid \exists l \in L(\mathcal{R}) \;:\; t|_p = \sigma(l)\}$. If $\mathcal{P}os_{\mathcal{R}}(t) = \varnothing$, we say that t is a normal form. A term t rewrites to s (at position p), written $t \to_{\mathcal{R}} s$ (or just $t \to s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ. The one-step rewrite relation for $\mathcal{R}$ is $\to$. If $t \to^* s$, then s is a reduct of t. A term is normalizing if it reduces to a normal form.

A TRS $\mathcal{R}$ is left-linear if $L(\mathcal{R})$ is a set of linear terms. Two rules $l \to r$ and $l' \to r'$ (whose variables have been possibly renamed to verify $\mathcal{V}ar(l) \cap \mathcal{V}ar(l') = \varnothing$) *overlap*, if there is a non-variable position $p \in \mathcal{P}os(l)$ and a most-general unifier σ such that $\sigma(l|_p) = \sigma(l')$. The pair $\langle \sigma(l)[\sigma(r')]_p, \sigma(r) \rangle$ is called a critical pair and is also called an overlay if $p = \Lambda$. A critical pair $\langle t, s\rangle$ is trivial if $t = s$. A TRS with critical pairs is called an overlapping TRS; left-linear, nonoverlapping TRSs are called orthogonal. It is called almost orthogonal if its critical pairs are trivial overlays. If it only has trivial critical pairs it is called weakly orthogonal.

3 Sequentiality

We introduce a new constant symbol Ω to represent arbitrary terms. Terms in $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{X})$ which we denote $\mathcal{T}_\Omega(\mathcal{F}, \mathcal{X})$) are called Ω-terms and they are used to denote prefixes of terms. Let $\mathcal{P}os_\Omega(t)$ be the set of Ω-positions of t, i.e., positions $p \in \mathcal{P}os(t)$ such that $t|_p = \Omega$. An ordering $\leq$ on Ω-terms is given as the least ordering which satisfies $\Omega \leq t$ for all t, and $f(t_1, \ldots, t_k) \leq f(s_1, \ldots, s_k)$ if $t_i \leq s_i$ for all $1 \leq i \leq k$. Thus, $t \leq s$ means that t is a prefix of s. We write $t \uparrow s$ if t and s are *compatible*, i.e., if there exists u such that $t \leq u$ and $s \leq u$. We denote by t_Ω the term t where all variables are replaced by Ω. An Ω-normal form is an Ω-term t such that $\mathcal{P}os_\mathcal{R}(t) = \varnothing$ and $\mathcal{P}os_\Omega(t) \neq \varnothing$. A *redex scheme* of a TRS $\mathcal{R}$ is a *lhs* of a rule $l \to r$ where all variables are replaced by Ω. Let $L_\Omega(\mathcal{R}) = \{l_\Omega \mid l \in L(\mathcal{R})\}$ be the set of redex schemes of $\mathcal{R}$. A *preredex* is an Ω-term π such that $\pi \leq l$ for some $l \in L_\Omega(\mathcal{R})$; a *proper* preredex is neither a redex scheme nor Ω. Let $L^<_\Omega(\mathcal{R}) = \{\pi \mid \exists l \in L_\Omega(\mathcal{R}), \pi < l\} - (L_\Omega(\mathcal{R}) \cup \{\Omega\})$ be the set of proper preredexes of $\mathcal{R}$.

The notion of sequentiality has been used to define normalizing rewriting strategies. Sequentiality is based on the notion of index. An Ω-position $p \in \mathcal{P}os_\Omega(t)$ of $t \in \mathcal{T}_\Omega(\mathcal{F}, \mathcal{X})$ is an index with respect to a predicate P on Ω-terms if, for every Ω-term s with $s \geq t$, $P(s)$ implies $s|_p \neq \Omega$ [KM91]. The set of indices of t with respect to P is denoted by $\mathcal{I}_P(t)$. A monotone predicate[2] P is sequential if for all $t \in \mathcal{T}_\Omega(\mathcal{F}, \mathcal{X})$, whenever $P(t)$ does not hold and there exists s such that $s \geq t$ and $P(s)$ holds, it follows that $\mathcal{I}_P(t) \neq \varnothing$. A TRS $\mathcal{R}$ is sequential if predicate $nf_\mathcal{R}$ (where $nf_\mathcal{R}(t)$ holds if and only if t has a normal form in $\mathcal{T}(\mathcal{F}, \mathcal{X})$) is sequential.

Both sequentiality of indices and TRSs are undecidable and several decidable approximations have been investigated. In the following, we recall two of them and justify our choice.

Strong sequentiality. Given a TRS $\mathcal{R}$, the reduction relation $\to_?$ (arbitrary reduction) on $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is defined by: $t \to_? s$ if there are $p \in \mathcal{P}os_\mathcal{R}(t)$ and $s' \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $s = t[s']_p$ [KM91]. A TRS $\mathcal{R}$ is *strongly sequential* [HL91, KM91] if predicate $nf_?$ is sequential (where $nf_?(t)$ holds if there exists an arbitrary reduction sequence $t \to^*_? s$ to some normal form $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$). Indices with respect to predicate $nf_?$ are said to be *strong* indices (and they are sequential indices, i.e., indices of $nf_\mathcal{R}$). The set of strong indices of a term t is denoted by $\mathcal{I}_s(t)$ (rather than $\mathcal{I}_{nf_?}(t)$).

For left-linear TRSs, strongly sequential indices can be effectively computed by using Ω-reduction: $t \to_\Omega s$ if $\exists p \in \mathcal{P}os(t)$ such that $t|_p \neq \Omega$, $t|_p \uparrow l$ for some $l \in L_\Omega(\mathcal{R})$, and $s = t[\Omega]_p$. The reduction relation $\to_\Omega$ is confluent and terminating for arbitrary TRSs [HL91, KM91]. Let $t{\downarrow_\Omega}$ be the $\to_\Omega$-normal form of t.

Proposition 1 [KM91, Lemma 4.8] *Let $\mathcal{R}$ be a left-linear TRS. Let $t \in \mathcal{T}_\Omega(\mathcal{F}, \mathcal{X})$ and $p \in \mathcal{P}os_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. Then, $p \in \mathcal{I}_s(t)$ if and only if $(t[\bullet]_p{\downarrow_\Omega})|_p = \bullet$.*

Left-linearity is essential for ensuring completeness of the technique. In the following example (and also in the remainder of the paper) letters a, b, c, ... denote constant symbols, whereas x, y, z denote variable symbols.

Example 2 *Consider the TRS $\mathcal{R}$:*

```
f(x,x) → g(a)
```

and $t =$ `f(g(Ω),a)`. Since `f(g(•),a)`$\downarrow_\Omega = \Omega$, according to Proposition 1, $1.1 \notin \mathcal{I}_s(t)$. However, since every $t' \geq t$ that has a normal form without Ω's must satisfy $t'|_{1.1} \neq \Omega$ (because t is a normal form), it follows that 1.1 is a strongly sequential index.

NV-sequentiality. Given a TRS $\mathcal{R} = (\mathcal{F}, R)$, we write $t \to_{nv} s$ if $\exists p \in \mathcal{P}os(t)$ and $l \to r \in R$ such that $t|_p \geq l_\Omega$ and $s = t[r']_p$ for some $r' \geq r_\Omega$. Note that, in contrast to $\to_?$, the *non-variable* part of the *rhs*'s of the rules is used to define the auxiliary reduction relation (this is the reason for the name – NV – of the approximation). Then, $nv(t)$ holds[3] if and only if $\exists s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $t \to^*_{nv} s$. Predicate nv is clearly monotone.

Definition 1 [Oya93] *A TRS is NV-sequential if $\mathcal{I}_{nv}(t) \neq \varnothing$ for every Ω-normal form t.*

We have the following.

Proposition 2 [Oya93] *Strong indices are nv-indices.*

We write $t \to_\omega s$ if $\exists p \in \mathcal{P}os(t)$ such that $t|_p \neq \Omega$, $t|_p \uparrow l_\Omega$ for some rule $l \to r$, and $s = t[r_\Omega]_p$. Oyamaguchi characterizes nv-indices as follows.

[2] Consider False $\leq$ True for booleans.

[3] In [Oya93] predicate nv is called *term*.

Lemma 1 [Oya93] *Let $t \in \mathcal{T}_\Omega(\mathcal{F},\mathcal{X})$ and $p \in \mathcal{P}os_\Omega(t)$. Then $p \notin \mathcal{I}_{nv}(t)$ if and only if there exist $q \in \mathcal{P}os(t)$, where $q < p$, and $s \in \mathcal{T}_\Omega(\mathcal{F},\mathcal{X}) - \{\Omega\}$ such that $t[\bullet]_p|_q \to_\omega^* s$ and $s \uparrow l$ for some $l \in L_\Omega(\mathcal{R})$.*

NVNF-sequentiality. NVNF-sequentiality is an extension of NV-sequentiality which is based on a new predicate *nvnf* which still uses the reduction relation $\to_{nv}$ introduced by Oyamaguchi. According to [NST95, Definition 3.2], *nvnf*(t) holds if $t \to_{nv}^* s$ for some $s \in \mathcal{T}(\mathcal{F},\mathcal{X})$ in normal form. Then, a TRS is NVNF-sequential if every Ω-normal form has an index w.r.t. *nvnf* [NST95, Definition 3.3].

Obviously, NVNF-sequential TRSs include NV-sequential TRSs. The inclusion is strict.

Example 3 *The following TRS*

```
f(a,b,x) → a        f(x,a,b) → c
f(b,x,a) → b        c        → c
```

is NVNF-sequential but not NV-sequential [NST95, Example 3.4].

Infinitary normalization and sequentiality. Root-needed reduction provides a suitable formal framework for the definition of root-normalizing, normalizing, and infinitary normalizing reduction sequences [Mid97]. A root-stable term (also called a head-normal form) is a term which cannot be reduced to a redex. A term is root-normalizing if it rewrites to a root-stable term. A redex in a term is *root-needed* if the redex (itself or one of its descendants) is reduced in each rewriting sequence leading to a root-stable term. Root-neededness is undecidable and it must be approximated. We have proved the following.

Theorem 1 [Luc98] *Let $\mathcal{R} = (\mathcal{F}, R)$ be an almost orthogonal TRS, $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ be non-root-stable, and $p \in \mathcal{P}os_\mathcal{R}(t)$. If $p \in \mathcal{I}_{nv}(t[\Omega]_p)$, then $t|_p$ is root-needed.*

Thus, *nv*-reduction strategies in NV-sequential TRSs can be used for infinitary normalization [Luc98].

4 Strong and NV-sequentiality in orthogonal CSs

Given a TRS $\mathcal{R} = (\mathcal{F}, R)$, we consider $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{f \mid f(l_1,\ldots,l_k) \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$. A *constructor* system (CS) is a TRS $\mathcal{R} = (\mathcal{F}, R)$ where for all $f(l_1,\ldots,l_k) \to r \in R$, we have that $l_1,\ldots,l_k \in \mathcal{T}(\mathcal{C},\mathcal{X})$.

Deciding whether an orthogonal TRS is strongly sequential is a co-NP complete problem [Dur95]. As noticed by Huet and Lévy [HL91], when considering CSs, things are simpler.

Proposition 3 [KM91] *An orthogonal CS $\mathcal{R}$ is strongly sequential if and only if $\forall\pi \in L_\Omega^<(\mathcal{R})$, $\mathcal{I}_s(\pi) \neq \varnothing$.*

Conditions in Proposition 3 are easy to check due to the following.

Proposition 4 [KM91] *Let $\mathcal{R}$ be a CS and $\pi \in L_\Omega^<(\mathcal{R})$. Then, $p \in \mathcal{I}_s(\pi)$ if and only if $\nexists l \in L_\Omega(\mathcal{R})$ such that $\pi[\bullet]_p \uparrow l$.*

For instance, by using these results, we prove that $\mathcal{R}$ in Example 1 is strongly sequential. Now we are ready to prove the following.

Theorem 2 *Every orthogonal NV-sequential CS is strongly sequential.*

Proof. By contradiction. Let $\mathcal{R}$ be an orthogonal NV-sequential CS which is not strongly sequential. By Proposition 3, there is $\pi \in L_\Omega^<(\mathcal{R})$ such that $\mathcal{I}_s(\pi) = \varnothing$. Since proper preredexes of orthogonal TRSs are Ω-normal forms, π is an Ω-normal form. Then, by NV-sequentiality, $\mathcal{I}_{nv}(\pi) \neq \varnothing$. Let $p \in \mathcal{I}_{nv}(\pi)$. Since $\pi \neq \Omega$, by Lemma 1 there is no $l \in L_\Omega(\mathcal{R})$ such that $\pi[\bullet]_p \uparrow l$. Thus, by Proposition 4, $p \in \mathcal{I}_s(\pi)$, and $\mathcal{I}_s(\pi) \neq \varnothing$. □

By [Oya93, Theorem 4.1], strongly sequential TRSs are NV-sequential. Thus, we have the following.

Corollary 1 *An orthogonal CS is NV-sequential if and only if it is strongly sequential.*

Currently, NVNF-sequential TRSs are the least class of approximations to sequentiality which properly includes NV-sequential TRSs. Example 3 shows that Theorem 2 does not hold for NVNF-sequential TRSs. Example 3 and Corollary 1 show that strongly (and NV-)sequential constructor systems are properly included into the class of NVNF-sequential systems. In [Jac96], more examples can be found showing that there is no greater class of approximations which collapse into a smaller one when considering CSs.

On the other hand, orthogonal NV-sequential (possibly non-constructor) TRSs properly include orthogonal strongly sequential TRSs.

Example 4 *Consider the orthogonal TRS $\mathcal{R}$ [Oya93]:*

$$\begin{array}{lll} \mathtt{f(f(a,x),f(b,y))} & \to & \mathtt{f(e,e)} \\ \mathtt{f(f(x,a),f(c,y))} & \to & \mathtt{f(e,e)} \\ \mathtt{f(d,d)} & \to & \mathtt{f(e,e)} \end{array}$$

Oyamaguchi showed that $\mathcal{R}$ is NV-sequential but not strongly sequential.

Hence, Theorem 2 and Corollary 1 do not generalise to non-constructor TRSs.

5 Overlapping CSs

Strong sequentiality was initially studied for orthogonal TRSs [HL91, KM91]. Strong sequentiality has been generalized to left-linear, possibly overlapping TRSs by Toyama[4] [Toy92, page 280].

In [JS94], Jouannaud and Sadfi addressed the decidability of strong sequentiality of left-linear TRSs which was claimed (without a detailed proof) by Toyama ([Toy92, Theorem 5.9]). As we will see, they in fact introduced a new notion of strong sequentiality instead.

5.1 Jouannud and Sadfi's notion of strong sequentiality

Jouannaud and Sadfi use Proposition 1 as the new *definition* of strong index.

Definition 2 [JS94] *Let $t \in \mathcal{T}_\Omega(\mathcal{F},\mathcal{X})$ and $p \in \mathcal{P}os_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. Then, p is said to be a strong index of t if $(t[\bullet]_p\downarrow_\Omega)|_p = \bullet$.*

However, Jouannaud and Sadfi's notion of strong sequentiality of a TRS is referred now to the existence of strong indices *in Ω-normal forms* only ([JS94, Definition 7]). In contrast to Toyama's, this definition does not exactly follow the original Huet and Lévy's definition of strong sequentiality. Thus, we call the new notion JS-strong sequentiality.

Definition 3 [JS94] *A left-linear TRS $\mathcal{R}$ is JS-strongly sequential if every Ω-normal form has a strong index.*

For orthogonal TRSs, we have the following.

Proposition 5 [HL91] *An orthogonal TRS $\mathcal{R}$ is (strongly) sequential if and only if $nf_\mathcal{R}$ (resp. $nf_?$) is sequential at every Ω-normal form.*

[4] Actually, the definition of (strong) index in [Toy92] is formally different: in fact, Toyama considers that a *redex position* p in a term t is an index of t if $(t[\bullet]_p\downarrow_\Omega)|_p = \bullet$. However, concerning its use for defining index reduction strategies, both definitions are equivalent.

Thus, strong sequentiality and JS-strong sequentiality coicide for orthogonal TRSs. This fact does not hold anymore for overlapping TRSs.

Example 5 *Consider the overlapping CS $\mathcal{R}$ [HL91]:*

$$\begin{array}{lllllll} \mathtt{f(a,b,x)} & \to & \mathtt{k} & \quad & \mathtt{c} & \to & \mathtt{a} \\ \mathtt{f(b,x,a)} & \to & \mathtt{k} & & \mathtt{c} & \to & \mathtt{b} \end{array}$$

$\mathcal{R}$ is not *sequential because the term $t = \mathtt{f(c,}\Omega\mathtt{,}\Omega\mathtt{)}$ has no index: Note that $s = \mathtt{f(c,}\Omega\mathtt{,a)} > \mathtt{f(c,}\Omega\mathtt{,}\Omega\mathtt{)}$ and $\mathtt{f(c,}\Omega\mathtt{,a)}$ has a normal form (without Ω's):*

$$\mathtt{f(\underline{c},}\Omega\mathtt{,a)} \to \underline{\mathtt{f(b,}\Omega\mathtt{,a)}} \to \mathtt{k}$$

but $s|_2 = \Omega$, i.e., 2 is not a sequential index of t. On the other hand, $s' = \mathtt{f(c,b,}\Omega\mathtt{)} > \mathtt{f(c,}\Omega\mathtt{,}\Omega\mathtt{)}$ and

$$\mathtt{f(\underline{c},b,}\Omega\mathtt{)} \to \underline{\mathtt{f(a,b,}\Omega\mathtt{)}} \to \mathtt{k}$$

but $s'|_3 = \Omega$, i.e., 3 is not a sequential index of t. Hence, t has no sequential index. Since $nf_\mathcal{R}(t)$ does not hold whereas both $nf_\mathcal{R}(s)$ and $nf_\mathcal{R}(s')$ hold, and t has no sequential index, $\mathcal{R}$ is not sequential. Thus, $\mathcal{R}$ is not strongly sequential either. However, as we show below (see Example 10), $\mathcal{R}$ is JS-strongly sequential.

5.2 Strong and NV-sequentiality of overlapping CSs

Jouannaud and Sadfi have proved that JS-strong sequentiality of left-linear TRSs is decidable [JS94]. Unfortunately, regarding CSs, the simple characterization of strong sequentiality of orthogonal CSs given by Proposition 3 in terms of the indices contained in the proper preredexes of the system does *not* extend to JS-strong sequentiality.

Example 6 *The left-linear, overlapping CS $\mathcal{R}$*

$$\begin{array}{lll} \mathtt{f(b,h(a))} & \to & \mathtt{a} \\ \mathtt{f(b,x)} & \to & \mathtt{a} \end{array}$$

is strongly sequential: it is left-normal[5] *and, according to [Toy92], left-normal, left-linear TRSs are JS-strongly sequential. However, the proper preredex $\mathtt{f(b,h(}\Omega\mathtt{))}$ has no strong index:*

$$\mathtt{f(b,h(}\bullet\mathtt{))} \to \Omega$$

As remarked by Jouannaud and Sadfi, proper preredexes in overlapping TRSs can be redexes also (for instance, consider the proper preredex $\mathtt{f(b,h(}\Omega\mathtt{))}$ in Example 6). For this reason, they introduce the notion of *atomic* preredex.

Definition 4 (Atomic preredex [JS94]**)** *Let $\mathcal{R}$ be a TRS. An atomic preredex is a proper preredex π satisfying: (1) For any redex scheme $l \in L_\Omega(\mathcal{R})$ compatible*

[5] A left-normal TRS is a TRS where the function and constant symbols occur to the left of variables in all left-hand sides [BN98].

with π, $\pi < l$; and (2) any strict subterm of π is not a preredex.

Let $L^A_\Omega(\mathcal{R})$ be the set of atomic preredexes of a TRS $\mathcal{R}$; note that $\Omega \notin L^A_\Omega(\mathcal{R})$. As remarked by Jouannaud and Sadfi, in contrast to proper preredexes, atomic preredexes are Ω-normal forms even for overlapping TRSs.

Remark 1 *For CSs, condition* (2) *in Definition 4 holds for all preredexes. Atomic preredexes of CSs are, then, completely characterized by* (1) *in Definition 4.*

Now, we can give a simple characterization of JS-strong sequentiality of left-linear CSs by using the notion of atomic preredex. First, we need the following.

Definition 5 [JS94] *A set $\{(p_i, s_i)\}_{i\in J}$ of pairs of positions and atomic preredexes is called a decomposition if (1) there is $i \in J$, $p_i = \Lambda$, and (2) for all $i \in J$ such that $p_i < p_k$ for some $k \in J$, there exists $j \in J$ such that $s_j|_{p_i/p_j} = \Omega$.*

The term $\tau(D)$ associated to a given decomposition D is defined inductively as follows [JS94]:

1. s if D is a singleton $\{(\Lambda, s)\}$,
2. $\tau((D-\{(\Lambda, s), (p_i, s_i)\}) \cup \{(\Lambda, s[s_i]_{p_i})\})$, if $(p_i, s_i) \in D$ and $s|_{p_i} = \Omega$.

An Ω-normal form t is called *decomposable* if there exists a decomposition D such that $t = \tau(D)$.

Example 7 *Consider the following TRS [JS94]:*

$$\begin{array}{ll} \texttt{f(h(h(x)),h(g(y)),g(z))} & \to \texttt{x} \\ \texttt{f(h(g(x)),h(h(y)),g(z))} & \to \texttt{x} \\ \texttt{h(g(g(x)))} & \to \texttt{x} \end{array}$$

Then, $\texttt{f(h(}\Omega\texttt{),h(h(}\Omega\texttt{)),g(}\Omega\texttt{))}$ *is decomposable into four atomic preredexes:*

$$\{(\Lambda, \texttt{f(}\Omega\texttt{,}\Omega\texttt{,g(}\Omega\texttt{))}), (1, \texttt{h(}\Omega\texttt{)}), (2, \texttt{h(}\Omega\texttt{)}), (2.1, \texttt{h(}\Omega\texttt{)})\}.$$

We have the following characterization of strong sequentiality of a TRS.

Proposition 6 [JS94] *A left-linear TRS $\mathcal{R}$ is JS-strongly sequential if and only if every decomposable term has a strong index.*

Remark 2 *Definition 4 does not exactly correspond to Jouannaud and Sadfi's. In [JS94, Definition 12], the authors first define the notion of* maximal atomic preredex *as maximal proper preredexes satisfying conditions (1) and (2) in Definition 4, and then they define an atomic preredex as a proper preredex which is smaller than or equal to a maximal atomic preredex. This style of definition can lead to problems when such maximal atomic preredexes do not exist; then, no atomic preredex can exist either. For instance, the following TRS $\mathcal{R}$:*

$$\begin{array}{l} \texttt{f(x,a,b)} \to \texttt{a} \\ \texttt{f(b,x,a)} \to \texttt{a} \\ \texttt{f(a,b,x)} \to \texttt{a} \end{array}$$

is a well-known example of non-strongly sequential TRS. According to Definition 4, $\mathcal{R}$ has an atomic preredex $\texttt{f(}\Omega\texttt{,}\Omega\texttt{,}\Omega\texttt{)}$*. However, according to [JS94], $\mathcal{R}$ has no atomic preredex. Thus, no term is decomposable and $\mathcal{R}$ would erroneously be considered as strongly sequential by using Proposition 6.*

According to Definition 2, we have the following property of strong indices in CSs that corresponds to Proposition 7.6 in [KM91] (whose proof also works for arbitrary CSs).

Proposition 7 [KM91] *Let $\mathcal{R}$ be a CS. Let $p \in \mathcal{I}_s(t)$ and s be such that $root(s) \in \mathcal{F}$ and $q \in \mathcal{I}_s(s)$. Then $p.q \in \mathcal{I}_s(t[s]_p)$.*

By using Proposition 6 (whose proof is correct for the notion of atomic preredex expressed by Definition 4), we prove the following.

Theorem 3 *A left-linear CS $\mathcal{R}$ is JS-strongly sequential if and only if $\forall\pi \in L^A_\Omega(\mathcal{R})$, $\mathcal{I}_s(\pi) \neq \varnothing$.*

Proof. If $\mathcal{R}$ is JS-strongly sequential, then, since every atomic preredex $\pi \in L^A_\Omega(\mathcal{R})$ is an Ω-normal form, by definition of JS-strong sequentiality $\mathcal{I}_s(\pi) \neq \varnothing$ and the conclusion follows.

Now we prove that whenever $\mathcal{I}_s(\pi) \neq \varnothing$ for each $\pi \in L^A_\Omega(\mathcal{R})$, every decomposable term t has a strong index. If t is decomposable, then there is a decomposition D such that $\tau(D) = t$. We proceed by induction. For the base case, we consider that $D = \{(\Lambda, t)\}$. Then t is an atomic preredex and $\mathcal{I}_s(t) \neq \varnothing$. On the other hand, if D is not a singleton, then $\tau(D) = \tau((D - \{(\Lambda, s), (p_i, s_i)\}) \cup \{(\Lambda, s[s_i]_{p_i})\})$ for some $(p_i, s_i) \in D$ such that $s|_{p_i} = \Omega$. By the induction hypothesis, $\mathcal{I}_s(s) \neq \varnothing$ and, by hypothesis, $\mathcal{I}_s(s_i) \neq \varnothing$. Let $q_i \in \mathcal{I}_s(s_i)$. If $p_i \in \mathcal{I}_s(s)$, then by Proposition 7, $p_i.q_i \in \mathcal{I}_s(s[s_i]_{p_i})$. On the other hand, if $p_i \notin \mathcal{I}_s(s)$, then we take an arbitrary $q \in \mathcal{I}_s(s)$. Note that $q \parallel p_i$. Then, by [KM91, Proposition 4.1(1)], $q \in \mathcal{I}(s[s_i]_{p_i})$. Thus, our hypothesis implies that every decomposable term has a strong index; now, by Proposition 6, $\mathcal{R}$ is JS-strongly sequential and, hence, the conclusion follows. □

Example 8 *The only atomic preredex of $\mathcal{R}$ in Example 6 is* $\texttt{f(}\Omega\texttt{,}\Omega\texttt{)}$*. It is easy to see that $1 \in \mathcal{I}_s($*$\texttt{f(}\Omega\texttt{,}\Omega\texttt{)}$*$)$. Thus, Theorem 3 proves JS-strong sequentiality of $\mathcal{R}$.*

In general, $L^{<}_{\Omega}(\mathcal{R}) \neq L^{A}_{\Omega}(\mathcal{R})$, even for orthogonal CSs.

Example 9 *Consider the TRS:*

```
f(a,b) → a
f(x,c) → a
```

Then, $\mathtt{f(a,}\Omega\mathtt{)}$ *is a proper preredex which is not atomic:* $\mathtt{f(a,}\Omega\mathtt{)} \uparrow \mathtt{f(}\Omega\mathtt{,c)}$ *but* $\mathtt{f(a,}\Omega\mathtt{)} \not< \mathtt{f(}\Omega\mathtt{,c)}$.

Therefore, dealing with orthogonal CSs, Theorem 3 can be considered as a refinement of Proposition 3.

Remark 3 *The atomic preredexes of an orthogonal strongly sequential CS $\mathcal{R}$ are just the inner (technically* branch*) nodes of the* definitional trees *that can be associated to $\mathcal{R}$ [HLM98]. Succeeding in associating such definitional trees to a CS implies that all inner nodes contain at least one index (the so-called* inductive position*, see [HLM98, Proposition 12]). Thus, Proposition 3 can also be seen as a different way to justify such decision procedure. See [HLM98] for further details about connections between strongly sequential systems and definitional trees.*

Now, we can show that JS-strong sequentiality differs from strong sequentiality.

Example 10 *Consider the CS $\mathcal{R}$ in Example 5. Each atomic preredex of $\mathcal{R}$ (namely* $\mathtt{f(}\Omega,\Omega,\Omega\mathtt{)}$, $\mathtt{f(a,}\Omega,\Omega\mathtt{)}$, *and* $\mathtt{f(b,}\Omega,\Omega\mathtt{)}$*), has a strong index. Thus, by Theorem 3, $\mathcal{R}$ is JS-strongly sequential*[6]*. However, as shown in Example 5, $\mathcal{R}$ is not strongly sequential.*

Concerning NV-sequentiality of left-linear, possibly overlapping CSs, we note that Oyamaguchi explicitly chooses the 'Ω-normal form' style for his initial definition and, in fact, the connections between strongly sequential and NV-sequential indices and between JS-strong and NV- sequentiality of TRSs remain valid for left-linear, possibly overlapping TRSs. Hence, by using Theorem 3, we can prove in the very same way as Theorem 2, the following.

Theorem 4 *Every left-linear, NV-sequential CS is JS-strongly sequential.*

Corollary 2 *A left-linear CS is NV-sequential if and only if it is JS-strongly sequential.*

[6]Actually this fact was already noticed by Huet and Lévy since in the discussion about properties of $\mathcal{R}$ in Example 5, they say that $\mathcal{R}$ 'is sequential at every Ω-normal form' ([HL91], page 420).

5.3 Two notions of strong sequentiality for overlapping TRSs

According to our previous discussion (see, e.g., Example 10), there are two different notions of strong sequentiality of left-linear (overlapping) TRSs in the literature. Note that the notion of strong index is the same in both approaches. The difference arises in Jouannaud and Sadfi's choice of restricting the attention to Ω-normal forms rather than arbitrary Ω-terms. Hence, strongly sequential TRSs are JS-strongly sequential. Moreover, Example 5 shows that this inclusion is strict.

After Jouannaud and Sadfi's work, Comon considered strong sequentiality of left-linear TRSs in its 'original' sense, i.e., as sequentiality of $nf_?$ [Com95, Com00]. In fact, he gave the missing proof of Toyama's decidability result that Jouannaud and Sadfi were looking for [Com00, Corollary 13].

As far as the author knows, these facts has not been noticed before and, surprisingly, all these works refer to each other as if they were discussing the same notion.

Some confusion has also arisen concerning NV and NVNF-sequentiality of left-linear TRSs. In [Com95] the author claims to provide the proof of decidability of NV-sequentiality of left-linear TRSs. However, his definition (see Section 4.3 in [Com95]) corresponds to $\mathcal{R}_{nv}$-sequentiality (where $\mathcal{R}_{nv}$ is a TRS with extra variables which is obtained by replacing variables in all right-hand sides of rules by new variables). In [Com00], the same author notices this point but now says that his results provide a proof of decidability of NVNF-sequentiality. However, the original definition of NVNF-sequentiality (of left-linear TRSs) does *not* correspond to $\mathcal{R}_{nv}$-sequentiality, since sequentiality of predicate $nf_{\mathcal{R}_{nv}}$ is required only on Ω-normal forms [Nag99, NST95]. Nagaya claims that decidability of NVNF-sequentiality has been proved by Comon (see [Nag99], page 16) whereas Comon only proves decidability of $\mathcal{R}_{nv}$-sequentiality (less general than NVNF-sequentiality). Fortunately, by following Comon's development, it is not difficult to see that, given a predicate P, predicate $\forall t \in \mathsf{NF}^{\Omega}_{\mathcal{R}}, (\exists s \in \mathcal{T}_{\Omega}(\mathcal{F},\mathcal{X}), P(s) \wedge t \leq s) \Rightarrow (P(t) \vee \exists p \in \mathcal{P}os(t), p \in \mathcal{I}_P(t))$ which expresses sequentiality of P restricted to Ω-normal forms (where $\mathsf{NF}^{\Omega}_{\mathcal{R}}$ denotes the set of Ω-normal forms of TRS $\mathcal{R}$) is also expressible in WSkS logic (see [Com00]) which extends Comon's results on decidability of $\mathcal{R}_{nv}$-sequentiality to NVNF-sequentiality too.

6 Index reduction strategies

The use of index reduction strategies in orthogonal TRSs is motivated by the following well-known result

[HL91]: *Index reduction is normalizing for orthogonal TRSs.* In particular, strong and *nv*-index reduction strategies are normalizing for orthogonal TRSs.

However, dealing with possibly overlapping TRSs, index reduction is *not* normalizing without any requirement of confluence as noticed by Toyama [Toy92]. Toyama defines a class of left-linear TRSs, which are the strongly sequential, root-balanced joinable TRSs for which reduction of strong indices is normalizing. A TRS is root-balanced joinable if all its critical pairs are root-balanced joinable. A critical pair $\langle t, s\rangle$ is root-balanced joinable if $t \xrightarrow{k}_r t'$ and $s \xrightarrow{k}_r t'$ for some t' and $k \geq 0$, where $\to_r$ is *root reduction*, i.e., $t \to_r s$ iff $t \to s$ and t is a redex[7], and $\xrightarrow{k}_r$ is the k-th iterate of $\to_r$-reduction.

Theorem 5 [Toy92] *Strong index reduction is normalizing for left-linear, strongly sequential, root-balanced joinable TRSs.*

Root-balanced joinable TRSs include orthogonal TRSs and also weakly orthogonal TRSs.

We further note that the identity between (JS-)strongly sequential and NV-sequential CSs does not mean that strong and *nv*-indices also coincide.

Example 11 *Consider the orthogonal CS $\mathcal{R}$:*

```
f(x,a) → c
g(a,x) → c
```

Note that t = `f(g(Ω,x),g(Ω,x))` *is an Ω-normal form. Position* 1.1 *corresponds to an nv-index, since we only have the following* $\to_\omega$*-reduction step:*

$$\mathtt{f(g(\bullet,x),}\underline{\mathtt{g(\Omega,x)}}\mathtt{)} \to_\omega \mathtt{f(g(\bullet,x),c)}$$

and `f(g(•,x),c)` *cannot be further reduced by* $\to_\omega$*. However,*

$$\mathtt{f(g(\bullet,x),}\underline{\mathtt{g(\Omega,x)}}\mathtt{)} \to_\Omega \underline{\mathtt{f(g(\bullet,x),\Omega)}} \to_\Omega \Omega$$

that is, $1.1 \notin \mathcal{I}_s(t)$*. Hence,* $\mathcal{I}_{nv}(t) \neq \mathcal{I}_s(t)$*.*

Therefore, strong index reduction and *nv*-index reduction strategies for CSs differ. Nagaya has investigated the properties of *nv*-index reduction strategies for left-linear TRSs [Nag99]. He has defined the class of NV-stable balanced joinable TRSs which properly includes root-balanced joinable TRSs. Nagaya proves that *nv*-index reduction is normalizing for left-linear NV-sequential, NV-stable balanced joinable TRSs. However, since he proves that strong index reduction is also normalizing for left-linear JS-strongly sequential[8], NV-stable balanced joinable TRSs, we conclude that, dealing with CSs (where NV-sequential and strongly sequential systems coincide, according to Corollary 2) there is no clear advantage in using *nv*-reduction instead of just strong index reduction.

Thus, concerning the implementation of normalizing and infinitary normalizing one-step rewriting strategies in CSs, no special effort is necessary for NV-sequentiality and available implementations based on exploiting strong sequentiality can be used instead.

7 Conclusion

Root-needed reduction provides an appropriate basis for normalization and infinitary normalization [Mid97]. In order to effectively define root-needed reduction strategies, root-neededness (in general undecidable) must be approximated. Among a number of already existing approximations to neededness, only the NV- and strong approximations have been proved correct as approximations to root-neededness, and hence useful to give a formal basis to the definition of both normalizing and infinitary normalizing strategies [Luc98]. Recently, this fact has been successfully used for the effective definition of context-sensitive rewriting strategies [Luc02]. This kind of strategies are based in forbidding reductions in fixed arguments of symbols and are used in a number of programming languages and reduction based systems, remarkably in Maude [CDEL+02].

In this paper, we have proved that strong sequentiality and NV-sequentiality coincide for orthogonal CSs (Corollary 1). Regarding left-linear CSs, we have proved that NV-sequentiality and Jouannaud and Sadfi's notion of strong sequentiality (called JS-strong sequentiality in this paper) are equivalent (Corollary 2). We have given a (correct and complete) characterization of left-linear JS-strongly sequential CSs in terms of the analysis of the atomic preredexes (Theorem 3). Since strong sequentiality and JS-strong sequentiality coincide for orthogonal TRSs, the result can also be used as a simpler test for orthogonal CSs which refines Huet and Levy's test of Proposition 3, since atomic preredexes are, in general, a strict subset of proper preredexes (also for orthogonal TRSs).

We have shown that strong index reduction in (JS-)strongly sequential CSs and *nv*-index reduction in NV-sequential CSs are essentially equivalent (Section 6). Hence, no special effort is necessary for the implementation of NV-sequentiality in CSs, since currently developed implementations based on (possibly restricted kinds of) strong sequentiality are appropiate. Our results also suggest that, regarding rewriting-based programming languages, where the use of CSs is not considered as a restriction but rather as an appro-

[7] This definition is taken from [Toy92]; the definition in [JS94] differs.

[8] The notion of strong sequentiality in Nagaya's work is Jouannaud and Sadfi's, see [Nag99, Definition 2.3.17].

priate discipline aimed at achieving a cleaner semantics and intuition of programs, strong sequentiality is the best framework to achieve correct and complete implementations, especially for lazy programming languages.

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