Termination of Fair Computations in Term Rewriting

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Abstract. The main goal of this paper is to apply rewriting termination technology —enjoying a quite mature set of termination results and tools— to the problem of proving automatically the termination of concurrent systems under fairness assumptions. We adopt the thesis that a concurrent system can be naturally modeled as a rewrite system, and develop a *reductionistic* theoretical approach to systematically transform, under reasonable assumptions, fair-termination problems into ordinary termination problems of associated relations, to which standard rewriting termination techniques and tools can be applied. Our theoretical results are combined into a practical *proof methodology* for proving fair-termination that can be automated and can be supported by current termination tools. We illustrate this methodology with some concrete examples and briefly comment on future extensions.

Keywords: Concurrent programming, fairness, term rewriting, program analysis, termination.

1 Introduction

This paper is about technology transfer. Our goal is to transfer a mature set of termination results and tools developed in recent years for term rewriting systems to prove termination of concurrent systems under fairness assumptions. This requires both adopting a certain theoretical stance about the modeling of concurrent systems, and developing new results and techniques to make the desired technology transfer possible. The theoretical stance in question is the thesis that *a concurrent system can be naturally modeled as a rewrite system*. This has by now been amply demonstrated to hold by theoretical approaches such as reduction semantics [BB92] and rewriting logic [Mes92], and by quite exhaustive studies showing that almost any imaginable concurrent system can be naturally modeled as a rewrite theory (see for example the survey [MM02]).

Once this theoretical stance is adopted, since fairness is a pervasive property of concurrent systems, needed to establish many properties of interest, the first thing required is to correctly express the fairness notion within the rewriting framework. In this regard, the early work of Porat and Francez [PF85, PF86], and the work of Tison for the ground fair termination case [Tis89], complemented by the more recent "localized fairness" notion in [Mes05] offer a good basis. As we explain in Section 7, other notions of fairness have also been proposed for rewrite systems, with other, quite different, motivations that make such notions inadequate for our purposes, namely, modeling concurrent systems. For concurrent systems, rewrite rules describe system transitions, and the notion of fair computation should require that if the rule is infinitely often enabled, then it is infinitely often taken.

Example 1. Consider the following TRS modeling a *scheduler* which is responsible for the distribution of processing in a concurrent operating system, where a number of processes `p` run independently.

```
[end]     exec(P) -> stop
[execute] schedule(cons(p,PS)) -> schedule(shift(exec(p),PS))
[remove]  schedule(cons(stop,PS)) -> schedule(PS)
[round]   schedule(cons(exec(P),PS)) -> schedule(shift(exec(P),PS))
[shift1]  shift(P,nil) -> cons(P,nil)
[shift2]  shift(P,cons(Q,PS)) -> cons(Q,shift(P,PS))
```

Processes are in one of three different states: *ready* (`p`), *running* (`exec(p)`), and *finished* (`stop`). A "round robin" fair scheduling strategy is to give each process a fixed amount of processing time and then shift the activity to the next one in a list of processes. If a process is ready, then it is executed (rule `execute`). If it is running, then the next one is taken (`round`). If the process stops, then it is removed from the system (`remove`). A running process `exec(p)` finishes when the rule `end` is applied. Although the system is clearly nonterminating, computations following the previous *fair* strategy will terminate. We will provide a formal proof of this claim later.

The situation in Example 1 cannot be modeled with other notions of fairness like the introduced in [KZ05] where fair rewriting computations can *only* be nonterminating, which makes any discussion of fair termination impossible.

The question that this paper then addresses, and presents partial answers to, is: how can rewriting termination techniques and tools be used to *automatically* prove the fair termination of a concurrent system? To the best of our knowledge, except for the quite restricted case of ground term rewriting systems for which Tison's tree automata techniques provide a decision procedure [Tis89], this precise question has not been previously posed or answered in the literature. Yet, we believe that, given the maturity of methods and tools for termination of rewrite systems, this is an important problem to attack, both theoretically and because of its many potential applications. The related question of finding general methods of proving fair termination of term rewriting systems has indeed been studied before, particularly by Porat and Francez [PF85, PF86]. However, their efforts followed the Floyd's classical approach, which uses predicates on states (in our setting, ground terms) to achieve termination (see [Fra86, Chapter 2] for a general description of this approach, and also [LPS81]). In particular, their characterization of fair termination of a rewrite system in terms of the compatibility of a well-founded ordering with all possible *full derivations* [PF86, Definition 9] does not lend itself to mechanization, since it suffers from the same problems as the Manna and Ness's classical termination criterion [MN70], namely, from the need to check *all* (infinitely many) full derivations, which makes automatic proofs of fair-termination quite hard.

Our approach is quite different. It is *reductionistic*, in the sense that *it seeks reasonable conditions under which fair-termination can be reduced to ordinary termination* of associated relations, for which standard rewriting termination techniques and tools can be applied. In Section 3, we show that the problem of proving (rule) fair-termination of a TRS $\mathcal{R}$ can be treated (without loss of generality) as the problem of proving fair-termination of $\mathcal{R}$ w.r.t. a subTRS $\mathcal{R}_F \subseteq \mathcal{R}$ of $\mathcal{R}$. If we take $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$, we show that fair-termination of $\mathcal{R}$ w.r.t. $\mathcal{R}_F$ can be proved by proving termination of the reduction relations $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ and $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ (Section 4). We prove that, if $\mathcal{R}_F$ is a single-rule TRS, then this is not only sufficient but also necessary for fair-termination of $\mathcal{R}$ w.r.t. $\mathcal{R}_F$. Then, in Section 5 we show how to translate such requirements into more standard termination problems, namely: proving or disproving termination, innermost termination, and relative termination of TRSs. Fortunately, methods for addressing such termination problems are currently available in existing termination tools like APROVE[3] and TPA[4], among others. Therefore, we get quite a practical approach for proving fair-termination of TRSs which clearly differs from more ad-hoc or restrictive approaches like the ones in [PF85, PF86, Tis89].

The results that we propose in this paper, although open to many extensions and generalizations, do indeed provide a quite practical *proof methodology* for proving fair-termination that can be automated and can be supported by current termination tools. In Section 5.4 we explain how our results can be synergistically combined into such a unified methodology, which offers different proof strategies to tackle a fair-termination problem. We show this methodology in action in proofs of concrete examples in Section 6. We consider the results obtained so far as encouraging, since they can allow proving fair-termination automatically. As we further discuss in Section 7, many extensions remain open as interesting research questions. However, our general methodology of reducing fair-termination to standard termination to try to make such proofs automatic is already a viable new methodology that we have put into practice using existing tools, and that we plan to incorporate into the Maude Termination Tool (MTT) [DLMMU04] and to further perfect as new results become available.

2 Preliminaries

Let $R \subseteq A \times A$ be a binary relation on a set A. We denote by R^+ the transitive closure of R and by R^* its reflexive and transitive closure. An R-sequence is a finite or countably infinite sequence (i.e., either $a_1, a_2, \ldots, a_n$ for some $n \in \mathbb{N}$, or $a_1, a_2, \ldots$) such that for a_i, a_{i+1} two consecutive elements in the sequence, we have $a_i \; R \; a_{i+1}$; we say that such a sequence begins with a_1 (if it is finite, we

[3] Available at `http://www-i2.informatik.rwth-aachen.de/AProVE`.

[4] Available at `http://www.win.tue.nl/tpa`.

also say that it *ends* with a_n). An element $a \in A$ is said to be an R-*normal form* if there exists no b such that $a\ R\ b$. The set of all R-normal forms is denoted by NF_R. We say that b is an R-normal form of a (written $aR^!b$) if $b \in \mathsf{NF}_R$ and $a\ R^* b$. We say that R is *terminating* iff there is no infinite sequence $a_1\ R\ a_2\ R\ a_3 \cdots$. Given binary relations R and S (on the same set A), we say that S *preserves* the R-*normal forms* if for each $a \in \mathsf{NF}_R$ and $b \in A$, $a\ S\ b$ implies that $b \in \mathsf{NF}_R$.

Throughout this paper, $\mathcal{X}$ denotes a countable set of variables, and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{f, g, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. The set of positions of a term t is $\mathcal{P}os(t)$. The subterm at position p of t is $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s.

A *rewrite rule* is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\mathcal{F}, R)$ with R a (possibly infinite) set of rewrite rules. A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ rewrites to s (at position p), written $t \xrightarrow{p}_{\mathcal{R}} s$ (or just $t \to s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $\rho : l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ. A TRS is *terminating* if $\to$ is terminating. The set of normal forms of $\mathcal{R}$ ($\mathcal{R}$-normal forms) is denoted by $\mathsf{NF}_{\mathcal{R}}$.

Given TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{S} = (\mathcal{F}, S)$, we denote by $\mathcal{R} \cup \mathcal{S}$ the TRS $(\mathcal{F}, R \cup S)$; also, we write $\mathcal{R} \subseteq \mathcal{S}$ to indicate that $R \subseteq S$.

The problem of proving termination of a TRS is equivalent to finding a well-founded, stable, and monotonic (strict) ordering $>$ on terms (i.e., a *reduction ordering*) which is *compatible* with the rules of the TRS, i.e., such that $l > r$ for all rules $l \to r$ of the TRS. Here, *monotonic* means that, for all k-ary symbol f, $i \in \{1, \ldots, k\}$, and $t, s, t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, whenever $t > s$, we have $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) > f(s_1, \ldots, t_{i-1}, s, \ldots, t_k)$. *Stable* means that, whenever $t > s$, we have $\sigma(t) > \sigma(s)$ for all terms t, s and substitutions σ.

3 Fairness and Fair Termination

The following definition is analogous to [PF85], but our formulation follows [Mes05]. Roughly speaking, an $\mathcal{R}$-sequence is fair (w.r.t. a subset of rules of $\mathcal{R}$) if each rule which is infinitely often enabled during the sequence is infinitely often taken.

Definition 1 (Rule fairness). *Given a TRS $\mathcal{R}$, we say that an $\mathcal{R}$-sequence $A : t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \cdots$ is* rule fair w.r.t. the rules in $\mathcal{R}_F \subseteq \mathcal{R}$ *(abbreviated* $\mathcal{R}_F$-fair*) if for all rules $\alpha : l \to r \in \mathcal{R}_F$, we have: If the set*

$$I^A_\alpha = \{i \in \mathbb{N} \mid \exists C_i, \sigma_i, p_i, s.t.\ t_i = C_i[\sigma_i(l)]_{p_i}\}$$

is infinite, then there is an infinite set $J^A_\alpha \subseteq I^A_\alpha$ such that, for all $j \in J^A_\alpha$, $t_j \to_{l \to r} t_{j+1}$.

As a simple consequence of Definition 1, finite $\mathcal{R}$-sequences are always fair w.r.t. any $\mathcal{R}_F \subseteq \mathcal{R}$. Also, all $\mathcal{R}$-sequences are fair w.r.t. $\mathcal{R}_F = \varnothing$.

Definition 2 (Rule fair-termination). *A TRS $\mathcal{R}$ is* fairly-terminating *w.r.t. $\mathcal{R}_F \subseteq \mathcal{R}$ if there is no infinite $\mathcal{R}_F$-fair $\mathcal{R}$-sequence. A TRS $\mathcal{R}$ is* rule fairly-terminating *if it is fairly-terminating w.r.t. $\mathcal{R}$ itself.*

Rule fair-termination coincides with Porat and Francez's [PF85] and the 'localized' definition w.r.t. a subset of rules $\mathcal{R}_F \subseteq \mathcal{R}$ is equivalent to [PF86, Definition 17]. Note that ordinary termination of TRSs is subsumed by Definition 2: take $\mathcal{R}_F = \varnothing$; then all $\mathcal{R}$-sequences are trivially fair w.r.t. $\mathcal{R}_F$, and $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$ if and only if $\mathcal{R}$ is terminating. And, clearly, termination of $\mathcal{R}$ impies rule fair-termination of $\mathcal{R}$. However, the opposite is not true: the system $\{$`a -> b`$,$ `a -> a`$\}$ is rule fairly-terminating but not terminating.

In contrast to ordinary termination, fair-termination is *not* preserved if some of the rules of the TRS are dismissed: there can be TRSs $\mathcal{R}$ which are $\mathcal{R}_F$-fairly-terminating for some $\mathcal{R}_F \subseteq \mathcal{R}$, whereas they are not $\mathcal{R}'_F$-fairly-terminating for a subset $\mathcal{R}'_F \subset \mathcal{R}_F$ of $\mathcal{R}_F$.

Example 2. Consider the following TRS $\mathcal{R}$ [PF85, Tis89]:

```
a -> f(a)
a -> b
```

As noticed by Tison, $\mathcal{R}$ is rule fairly-terminating (i.e., fairly-terminating w.r.t. $\mathcal{R}$ itself). Let $\mathcal{R}_F$ be the subTRS of $\mathcal{R}$ consisting of the first rule (then take $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$). The following infinite $\mathcal{R}$-sequence (as usual, we underline the contracted redex):

$$\underline{\mathtt{a}} \to_{\mathcal{R}_F} \mathtt{f(\underline{a})} \to_{\mathcal{R}_F} \mathtt{f(f(\underline{a}))} \to_{\mathcal{R}_F} \cdots$$

is $\mathcal{R}_F$-fair. This shows that $\mathcal{R}$ is not $\mathcal{R}_F$-fairly-terminating.

The key observation is that, given $\mathcal{R}_F, \mathcal{R}'_F \subseteq \mathcal{R}$, the set of $\mathcal{R}_F \cup \mathcal{R}'_F$-fair sequences is the *intersection* of the sets of $\mathcal{R}_F$-fair and $\mathcal{R}'_F$-fair sequences. Therefore, we have the following obvious sufficient condition in the other direction.

Proposition 1. *A TRS $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F \subseteq \mathcal{R}$ if there is a subset $\mathcal{R}'_F \subset \mathcal{R}_F$, such that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}'_F$.*

The subset $\mathcal{R}'_F$ in Proposition 1 can be a *single* rule. For instance, Tison observes that $\mathcal{R}$ in Example 2 is rule fairly-terminating thanks to the rule `a -> b`. As we shall see below, this is a specially interesting case. The system in Example 1, however, is $\mathcal{R}_F$-fairly-terminating provided that $\mathcal{R}_F$ contains all three rules `end`, `execute`, and `remove`. It is easy to see that the absence of one of them destroys fair-termination. Proposition 1 will be used later.

4 Reducing Fair Termination to Termination

Termination analysis has recently experimented a remarkable development in the term rewriting community, leading to the birth of a new generation of promising methods, tools, and technology transfer. An important goal of this paper is giving an appropriate theoretical basis for fair-termination on which machine-implementable fair-termination techniques can be based. In this section, we investigate how to reduce a proof of fair-termination to the problem of proving termination of particular (combinations of) reduction relations.

Intuitively, a sufficient condition for $\mathcal{R}_F$-fair-termination of a TRS $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ is that: (1) there is no infinite $\mathcal{R}$-sequence performing an infinite number of $\mathcal{R}_F$-steps, and (2) every infinite $\mathcal{S}$-sequence contains an $\mathcal{R}_F$-redex. The first condition corresponds to the termination of the relation $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ (which implies termination of $\mathcal{R}_F$). The second condition can be captured as the termination of the relation $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$. Note, however, that they are not equivalent. For instance, for $\mathcal{S} = \{$`a -> a`$,$`b -> a`$\}$ and $\mathcal{R}_F = \{$`a -> b`$\}$ we have that $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ is not terminating, but (2) holds. Theorem 1 below formalizes this intuition. In order to prove it, we first need the following.

Proposition 2. *Let $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ be a TRS such that $\mathcal{R}_F$ is finite and $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ is terminating. If $\mathcal{R}$ is not fairly-terminating w.r.t. $\mathcal{R}_F$, then for each infinite $\mathcal{R}_F$-fair $\mathcal{R}$-sequence A there is a rule $\alpha : l \to r \in \mathcal{R}_F$ for which I^A_α is infinite.*

Proof. We proceed by contradiction. If $\mathcal{R}$ is not fairly-terminating w.r.t. $\mathcal{R}_F$, then there is an infinite $\mathcal{R}_F$-fair $\mathcal{R}$-sequence A. Assume that there exists one such sequence A such that for all rules $\alpha : l \to r$ in $\mathcal{R}_F$, I^A_α is finite. Then, since $\mathcal{R}_F$ is finite, A can be written as follows: $A : t_1 \to^*_{\mathcal{R}} t_n \to_{\mathcal{S}} t_{n+1} \to_{\mathcal{S}} \cdots$ where the terms t_i contain no $\mathcal{R}_F$-redex for $i \geq n$. Then, those t_i are $\mathcal{R}_F$-normal forms. Since $t \to^!_{\mathcal{R}_F} t$ for any $\to_{\mathcal{R}_F}$-normal form t, we can write the subsequence of A starting from t_n as follows: $t_n \to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}} t_{n+1} \to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}} \cdots$ This contradicts the termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$. □

Theorem 1. *A TRS $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ with $\mathcal{R}_F$ finite is fairly-terminating w.r.t. $\mathcal{R}_F$ if $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ and $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ are terminating.*

Proof. Assume that $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ and $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ are terminating, and that $\mathcal{R}$ is not fairly-terminating w.r.t. $\mathcal{R}_F$. Then there is an infinite $\mathcal{R}_F$-fair $\mathcal{R}$-sequence A. By Proposition 2, there is a rule $\alpha : l \to r \in \mathcal{R}_F$ such that I^A_α is infinite. Since, by $\mathcal{R}_F$-fairness, J^A_α is infinite, A can be written as follows:

$$A : t_1 \;\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}\; t_{j_1+1} \;\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}\; t_{j_2+1} \;\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}\; \cdots$$

which contradicts termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$. □

The following example, however, shows that Theorem 1 does *not* provide a complete method for proving rule fair termination.

Example 3. Consider the following TRS $\mathcal{R}$ [PF85]:

```
a -> f(a)          g(a,b) -> c          a -> g(a,b)
```

which is rule fairly-terminating. It is not difficult to see that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F \subset \mathcal{R}$ given by the two rightmost rules above. Since $\mathcal{R}_F$ is *not* terminating, $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ is nonterminating. Therefore, Theorem 1 cannot be used to prove fair termination of $\mathcal{R}$ w.r.t. $\mathcal{R}_F$, even though $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$ and $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ is terminating.

Hence, termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ (alone) is not a necessary condition for fair-termination of $\mathcal{R}$ w.r.t. $\mathcal{R}_F$. Similarly, one could see that termination of $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ is *not* a necessary condition either. However, when $\mathcal{R}_F$ is a single rule TRS, we have the following characterization.

Theorem 2. *Let $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ and $\mathcal{R}_F$ be a single rule TRS. Then, $\mathcal{R}$ is $\mathcal{R}_F$-fairly-terminating if and only if $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ and $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ are terminating.*

Proof. The ($\Leftarrow$) part follows by Theorem 1. To prove the ($\Rightarrow$) part, we reason by contradiction and assume that either $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ or $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ are nonterminating. If $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ is nonterminating, then there is an infinite sequence: $A : t_1 \to_{\mathcal{S}}^{*} t_1' \to_{\mathcal{R}_F} t_2 \to_{\mathcal{S}}^{*} t_2' \to_{\mathcal{R}_F} \cdots$ which (by $\mathcal{R}_F$ containing only one rule) is $\mathcal{R}_F$-fair, thus contradicting $\mathcal{R}_F$-fair termination of $\mathcal{R}$. If $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ is nonterminating, then there is an infinite sequence $t_1 \to_{\mathcal{R}_F}^{!} t_1' \to_{\mathcal{S}} t_2 \to_{\mathcal{R}_F}^{!} t_2' \to_{\mathcal{S}} \cdots$ which, since $\mathcal{R}_F$ contains only one rule, is $\mathcal{R}_F$-fair: note that either t_i contains no $\mathcal{R}_F$-redex (and then $t_i' = t_i$) or t_i is normalized by $\mathcal{R}_F$ (hence all $\mathcal{R}_F$-redexes in t_i are contracted). $\square$

5 Proving Fair-Termination

According to Theorem 1, if we prove termination of both $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ and $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$, then fair-termination of $\mathcal{R} = \mathcal{S} \cup \mathcal{R}_F$ follows.

Note that given two reduction relations $\to_1$ and $\to_2$, the (non)termination of $\to_2^{*} \circ \to_1$ and $\to_1^{!} \circ \to_2$ do not have any (easy) connection: let $\to_1$ and $\to_2$ be relations on $A = \{a, b, c\}$ such that $a \to_1 b$ and $c \to_2 c$ are the only components of the respective relations. Then, $\to_2^{*} \circ \to_1 = \to_1$ is terminating but $\to_1^{!} \circ \to_2$ is not terminating: $c \to_1^{!} c \to_2 c \to_1^{!} c \to_2 \cdots$. On the other hand, $\to_2^{!} \circ \to_1$ is terminating (since $\to_2^{!} = \{(a, a), (b, b)\}$, we have $\to_2^{!} \circ \to_1 = \to_1$), but $\to_1^{*} \circ \to_2 \supseteq \to_2$ is not terminating. Thus, in the following, we consider how to address these two (more standard) termination problems in more detail.

5.1 Termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$

Given binary relations $\to_1$ and $\to_2$ on an abstract set A, $\to_1$ is called *relatively noetherian* (or better *relatively terminating*) with respect to $\to_2$ if every infinite $\to_1 \cup \to_2$-derivation contains only finitely many $\to_1$-steps (see [Ges90, Section 2.1], although the notion goes back to Klop: see also [Klo92, Exercise 2.0.8(11)]).

In his PhD thesis [Ges90], A. Geser has investigated relative termination. In our setting, this notion is interesting due to the following result.

Proposition 3. [Ges90] *Let $\to_1$ and $\to_2$ be binary relations. Then, $\to_2^{*} \circ \to_1$ is terminating if and only if $\to_1$ is relatively terminating with respect to $\to_2$.*

Thus, according to this result, termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ can be investigated as the relative termination of $\mathcal{R}_F$ w.r.t. $\mathcal{S}$. Fortunately, there are even automatic tools which can be used to prove relative termination of TRSs.

Example 4. Consider the TRS $\mathcal{R}$ in Example 2. Let $\mathcal{R}_F$ be the subTRS consisting of the rule `a -> b` and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. Now, TPA can be used to prove termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$. Consider again the system $\mathcal{R}$ in Example 1 with $\mathcal{R}_F$ consisting of the rules `end`, `execute`, and `remove` and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. We have used TPA to obtain an automatic proof of termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$.

5.2 Termination of $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$

Termination of $\to_2^{!} \circ \to_1$ for binary relations $\to_1$ and $\to_2$ can also be investigated as *relative* termination of $\to_1$ w.r.t. $\to_2$.

Proposition 4. *Let A be a set and $\to_1, \to_2 \subseteq A \times A$ be binary relations. If $\to_1$ is relatively terminating w.r.t. $\to_2$, then $\to_2^{!} \circ \to_1$ is terminating.*

PROOF. Since relative termination of $\to_1$ w.r.t. $\to_2$ is equivalent to termination of $\to_2^{*} \circ \to_1$ (Proposition 3) and, since $\to^{!} \subseteq \to^{*}$ for all binary relation $\to$, termination of $\to_2^{*} \circ \to_1$ implies termination of $\to_2^{!} \circ \to_1$. □

Since termination of $\to_{\mathcal{R}_F}^{*} \circ \to_{\mathcal{S}}$ implies termination of $\mathcal{S}$ and termination of $\to_{\mathcal{S}}^{*} \circ \to_{\mathcal{R}_F}$ (which is also required) implies termination of $\mathcal{R}_F$, this means that both $\mathcal{R}_F$ and $\mathcal{S}$ must be terminating (at least as separate TRSs) which is quite a restrictive setting. The following results are helpful to prove termination of $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$.

Proposition 5. *Let $\mathcal{R}$ and $\mathcal{S}$ be two TRSs. Let $\mathcal{S}' = \{l \to r \in \mathcal{S} \mid l \in \mathsf{NF}_{\mathcal{R}}\}$. Then, $\to_{\mathcal{R}}^{!} \circ \to_{\mathcal{S}'}$ is terminating if and only if $\to_{\mathcal{R}}^{!} \circ \to_{\mathcal{S}}$ is terminating.*

PROOF. By definition of $\mathcal{S}'$ and $\to_{\mathcal{R}}^{!}$, we have $(\to_{\mathcal{R}}^{!} \circ \to_{\mathcal{S}}) = (\to_{\mathcal{R}}^{!} \circ \to_{\mathcal{S}'})$. □

Example 5. Consider the TRS $\mathcal{R}$ in Example 2 with $\mathcal{R} = \mathcal{R}_F \cup \mathcal{S}$ as in Example 4. Since $\mathcal{S}'$ computed as in Proposition 5 is empty, $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ is terminating.

Consider again the TRS in Example 1 with $\mathcal{R}_F$ and $\mathcal{S}$ as in Example 4. The use of Proposition 5 produces a simpler version $\mathcal{S}'$ of $\mathcal{S}$, which consists of the rules `shift1` and `shift2`. Since $\mathcal{R}_F \cup \mathcal{S}'$ can be proved terminating (by using, e.g., APROVE), we have that $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}'}$ is clearly terminating. By Proposition 5, $\to_{\mathcal{R}_F}^{!} \circ \to_{\mathcal{S}}$ is also terminating.

Proposition 6. *Let A be a set and $\to_1, \to_2 \subseteq A \times A$ be binary relations. If $\to_2$ is terminating and preserves the $\to_1$-normal forms, then $\to_1^{!} \circ \to_2$ is terminating.*

PROOF. If $\to_1^{!} \circ \to_2$ is nonterminating, then there is an infinite sequence

$$t = t_1 \to_1^{!} t_1' \to_2 t_2 \to_1^{!} t_2' \to_2 \cdots$$

and since $\to_2$ preserves the $\to_1$-normal forms, we can then extract the infinite sequence $t = t_1' \to_2 t_2 \to_2 \cdots$ which contradicts termination of $\to_2$. □

The following example shows the limitations of this approach.

Example 6. Consider the following TRS $\mathcal{R}$:

```
f(a) -> a
f(X) -> f(a)
```

Let $\mathcal{R}_F$ be the subTRS of $\mathcal{R}$ consisting of the first rule and $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$. It is not possible to apply the results in this section to prove termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ (note that $\mathcal{S}$ is nonterminating and the *lhs* `f(X)` is an $\mathcal{R}_F$-normal form).

In the following section, we introduce a transformation for proving termination of $\to^!_{\mathcal{R}} \circ \to_{\mathcal{S}}$ for arbitrary TRSs $\mathcal{R}$ and $\mathcal{S}$.

5.3 Termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ by Transformation

Given TRSs $\mathcal{R}_1$ and $\mathcal{R}_2$, our idea here is to implement a 'distributed' computation by performing as many $\to_{\mathcal{R}_1}$-steps as possible (thus obtaining an $\mathcal{R}_1$-normal form) followed by a *single* $\to_{\mathcal{R}_2}$-step. Inspired by the transformations in [GM04] (which have been developed for a completely different purpose), our transformation keeps track of each *single* reduction step issued by $\mathcal{R}_2$. This is achieved by shifting a single symbol `active` to (non-deterministically) reach the position where a redex is placed. The application of a rewrite rule changes `active` into `mark`, which is propagated upwards through the term in order to be replaced by a new symbol `active` that enables new reduction steps. Given a TRS $\mathcal{R} = (\mathcal{F}, R)$, the TRS $\mathcal{U}_{\mathcal{R}} = (\mathcal{F} \cup \{\mathtt{active}, \mathtt{mark}, \mathtt{top}\}, U)$ consists of the following rules: for all $l \to r \in R$, $f \in \mathcal{F}$ such that $k = ar(f) > 0$, and $i \in \{1, \ldots, ar(f)\}$,

$$\begin{aligned} \mathtt{active}(l) &\to \mathtt{mark}(r) \\ \mathtt{active}(f(x_1, \ldots, x_i, \ldots, x_k)) &\to f(x_1, \ldots, \mathtt{active}(x_i), \ldots, x_k) \\ f(x_1, \ldots, \mathtt{mark}(x_i), \ldots, x_k) &\to \mathtt{mark}(f(x_1, \ldots, x_i, \ldots, x_k)) \\ \mathtt{top}(\mathtt{mark}(x)) &\to \mathtt{top}(\mathtt{active}(x)) \end{aligned}$$

We are actually interested in the *union* $\mathcal{R}_1 \cup \mathcal{U}_{\mathcal{R}_2}$ of $\mathcal{R}_1$ and $\mathcal{U}_{\mathcal{R}_2}$. In order to ensure that before starting the application of a rule marked with `active` (which belongs to $\mathcal{R}_2$), the argument of `mark` is in $\mathcal{R}_1$-normal form, we use *innermost* rewriting. We have the following:

Theorem 3. *Let $\mathcal{R}_1 = (\mathcal{F}, R_1)$ be a confluent and innermost terminating TRS and $\mathcal{R}_2 = (\mathcal{F}, R_2)$ be a TRS. If $\mathcal{R}_1 \cup \mathcal{U}_{\mathcal{R}_2}$ is innermost terminating, then $\to^!_{\mathcal{R}_1} \circ \to_{\mathcal{R}_2}$ is terminating.*

PROOF. By contradiction. Assume that $\to^!_{\mathcal{R}_1} \circ \to_{\mathcal{R}_2}$ is nonterminating. Then, there is an infinite sequence $t = t_1 \to^!_{\mathcal{R}_1} s_1 \to_{\mathcal{R}_2} t_2 \to^!_{\mathcal{R}_1} s_2 \to_{\mathcal{R}_2} \cdots$ starting from a term t. We show that there is an innermost counterpart in $\mathcal{R}_1 \cup \mathcal{U}_{\mathcal{R}_2}$ starting from $\mathtt{top}(\mathtt{mark}(t))$:

1. Since $\mathcal{R}_1$ is innermost terminating, there is s'_1 such that $t_1 \xrightarrow{i}{}^!_{\mathcal{R}_1} s'_1$; by confluence, $s'_1 = s_1$. Thus, we have $\mathtt{top}(\mathtt{mark}(t_1)) \xrightarrow{i}{}^!_{\mathcal{R}_1} \mathtt{top}(\mathtt{mark}(s_1))$. Furthermore, $\mathtt{top}(\mathtt{mark}(t_1)) \xrightarrow{i}{}^!_{\mathcal{R}_1 \cup \mathcal{U}_{\mathcal{R}_2}} \mathtt{top}(\mathtt{mark}(s_1))$.

2. Since s_1 is an $\mathcal{R}_1$-normal form, there is only one reduction step which can be issued on $\texttt{top(mark(}s_1\texttt{))}$, i.e., $\texttt{top(mark(}s_1\texttt{))}\xrightarrow{i}_{\mathcal{R}_1\cup\mathcal{U}_{\mathcal{R}_2}}\texttt{top(active(}s_1\texttt{))}$.
3. Finally, we have that $\texttt{top(active(}s_1\texttt{))} \xrightarrow{i}{}^{*}_{\mathcal{R}_1\cup\mathcal{U}_{\mathcal{R}_2}}\texttt{top(mark(}s_2\texttt{))}$. The need of considering the rules in $\mathcal{R}_1$ demands some further explanation. Since s_1 is an $\mathcal{R}_1$-normal form, all steps issued by the group of rules

$$\texttt{active(}f(x_1,\ldots,x_i,\ldots,x_k)) \to f(x_1,\ldots,\texttt{active(}x_i),\ldots,x_k)$$

which put symbol `active` deeper and deeper (until reaching the position of the $\mathcal{R}_2$-redex in s_1) are clearly innermost. After issuing the reduction step by using a rule $\texttt{active(}l) \to \texttt{mark(}r)$ for some $l \to r \in \mathcal{R}_2$, new $\mathcal{R}_1$-redexes can appear *below* symbol `mark` which signals the position of the recently contracted redex. The innermost reduction sequence could need to continue, then, by issuing $\mathcal{R}_1$-steps. After this partial innermost $\mathcal{R}_1$-normalization, a rule $f(x_1,\ldots,\texttt{mark(}x_i),\ldots,x_k) \to \texttt{mark(}f(x_1,\ldots,x_i,\ldots,x_k))$ would eventually apply as the only (innermost!) reduction step, to push up the symbol `mark`. These interleaved process would continue until putting `mark` immediately below `top`, having s_2 (in $\mathcal{R}_1$-normal form!) as the only argument.

This contradicts innermost termination of $\mathcal{R}_1 \cup \mathcal{U}_{\mathcal{R}_2}$. □

In our setting, we use Theorem 3 with $\mathcal{R}_1 = \mathcal{R}_F$ and[5] $\mathcal{R}_2 = \mathcal{S}$. In practice, checking innermost termination of $\mathcal{R}_F$ is not necessary if we have already proved that $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ is terminating because this implies termination of $\mathcal{R}_F$.

Example 7. Consider $\mathcal{R}$, $\mathcal{R}_F$ and $\mathcal{S}$ as in Example 6. Termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ can be proved with TPA. Regarding termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$, the transformed system TRS $\mathcal{R}_F \cup \mathcal{U}_{\mathcal{S}}$:

```
f(a) -> a
active(f(X)) -> mark(f(a))
active(f(X)) -> f(active(X))
f(mark(X)) -> mark(f(X))
top(mark(X)) -> top(active(X))
```

is innermost terminating (although we were not able to obtain an automatic proof). Note that $\mathcal{R}_F$ is clearly confluent. Therefore, by Theorem 3, we conclude termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$. Thus, the system $\mathcal{R}$ is fairly-terminating.

5.4 A Methodology for Proving Fair-Termination as Termination

PROBLEM 1: Given a TRS $\mathcal{R}$ and a finite subTRS $\mathcal{R}_F \subseteq \mathcal{R}$, is $\mathcal{R}$ fairly-terminating w.r.t. $\mathcal{R}_F$? We have two lines of attack:

1. *Prove termination of* $\mathcal{R}$: If $\mathcal{R}$ is terminating, then $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$.

[5] The tool MU-TERM provides an implementation of Giesl and Middeldorp's transformation from which $\mathcal{U}_{\mathcal{S}'}$ is easily obtained. MU-TERM is available on `http://www.dsic.upv.es/~slucas/csr/termination/muterm`.

2. If $\mathcal{R}_F$ is not terminating, then *look for a terminating subset* $\mathcal{R}'_F \subset \mathcal{R}_F$ *of* $\mathcal{R}_F$. By Proposition 1 we can change $\mathcal{R}_F$ be the selected $\mathcal{R}'_F$ and go to Problem 2 below to try to prove the new configuration of the problem.

PROBLEM 2: Given a TRS $\mathcal{R}$ and a finite and terminating subTRS $\mathcal{R}_F \subseteq \mathcal{R}$, is $\mathcal{R}$ fairly-terminating w.r.t. $\mathcal{R}_F$?

With $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$, according to Theorem 1, we try to prove termination of both $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ and $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$:

1. Prove the *relative termination of* $\mathcal{R}_F$ *w.r.t.* $\mathcal{S}$ (see Proposition 3). Termination tools like TPA can also be used to obtain an automatic proof.
2. Prove termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$: first, restrict the TRS $\mathcal{S}$ to $\mathcal{S}' \subseteq \mathcal{S}$ as indicated in Proposition 5. Now, we can prove termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}'}$ by using one of the following methods:
   (a) If $\mathcal{R}_F \cup \mathcal{S}'$ is terminating, then $(\to_{\mathcal{R}_F} \cup \to_{\mathcal{S}'})^+$ is terminating and therefore $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}'} \subseteq (\to_{\mathcal{R}_F} \cup \to_{\mathcal{S}'})^+$ also is.
   (b) If $\mathcal{S}'$ is terminating, then
      i. If $\mathcal{S}'$ preserves the $\mathcal{R}_F$-normal forms, then by Proposition 6, termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}'}$ follows.
      ii. Prove the relative termination of $\mathcal{S}'$ w.r.t. $\mathcal{R}_F$. By Proposition 4, this implies termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}'}$.
   (c) Otherwise, prove innermost termination of the union of $\mathcal{R}_F$ and the transformed TRS $\mathcal{U}_{\mathcal{S}'}$. If $\mathcal{R}_F$ is confluent, by Theorem 3 termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}'}$ follows.

PROBLEM 3: Is a TRS $\mathcal{R}$ rule fairly-terminating? We have two lines of attack:

1. *Prove termination of* $\mathcal{R}$: If $\mathcal{R}$ is terminating, then $\mathcal{R}$ is rule fairly-terminating.
2. According to Proposition 1, we can look for a subTRS $\mathcal{R}_F$ such that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$ (thus reducing to Problems 1 and 2).

Fortunately, the previous termination problems (proving termination, innermost termination, and relative termination of TRSs) are currently supported by existing termination tools like APROVE and TPA, among others.

6 Applications

In this section, we describe two more practical (still simple) examples of nonterminating systems which are fairly-terminating and show how to formally prove this property using our results and the methodology of Section 5.4.

Lottery

Consider the following scenario: a lottery where a finite number of balls are rolling inside a container assumed here to be circular. Eventually, a ball will be removed to pick a number and, of course, the repeated extraction of balls will make the whole process terminating. The following TRS can be used to model this process:

```
[extract]   cons(X,XS) -> XS
[shift]     cons(X,cons(Y,XS)) -> cons(Y,snoc(XS,X))
[circular1] snoc(nil,X) -> cons(X,nil)
[circular2] snoc(cons(X,XS),Y) -> cons(X,snoc(XS,Y))
```

Here, $\mathcal{R}_F$ consists of the rule `extract`, which represents the extraction of a ball. The remaining rules (`shift`, `circular1` and `circular2`) are collected into a nonterminating TRS $\mathcal{S}$ which represents a finite list whose elements are shifted in a circular fashion over and over again.

Let us prove that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$. According to Theorem 2, we *have* to prove that both $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$ and $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}}$ are terminating. Regarding termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$, by Proposition 3 this is equivalent to proving that $\mathcal{R}_F$ is relatively terminating with respect to $\mathcal{S}$. We have used TPA to obtain an automatic proof of this. Regarding termination of $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}}$, we can use Proposition 5 to obtain a subTRS $\mathcal{S}'$ of $\mathcal{S}$ which only contains `circular1`. By Proposition 5, termination of $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}}$ is equivalent to termination of $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}'}$. The TRS $\mathcal{S}'$ is obviously terminating. Since $\mathcal{R}_F \cup \mathcal{S}'$ is also terminating, $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}'}$ is terminating and $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$.

Noisy channel

Consider the following scenario: there are three agents A, B, and C. Agents A and B have to perform tasks a and b (respectively) in a distributed fashion. Agent C receives information about their completion through a two-component channel. Agent A (resp. B), writes "a", (resp. "b") on the corresponding channel to communicate to C that his/her task has been finished. Once the tasks performed by A and B have both terminated, C closes the channel. However, the channel is *noisy* in such a way that, when both values are on it, they can get lost. Thus, both A and B may have to repeat their respective signals before the channel is closed. The following TRS can be used to model this process:

```
[A]    [null,Y] -> [a,Y]
[B]    [X,null] -> [X,b]
[C]    [a,b] -> done
[loss] [a,b] -> [null,null]
```

The key point here is that if rule `C` is fair, then the system is terminating. Thus, we consider $\mathcal{R}_F$ consisting of rule `C`.

Let us prove that $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$. Let $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$, i.e., $\mathcal{S}$ contains the rules `A`, `B` and `loss` (and it is nonterminating). According to Theorem 2, we have to prove that both $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$ and $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}}$ are terminating. Regarding termination of $\to_{\mathcal{S}}^* \circ \to_{\mathcal{R}_F}$, by Proposition 3 this is equivalent to proving that $\mathcal{R}_F$ is relatively terminating with respect to $\mathcal{S}$. Again, we have used TPA to obtain an automatic proof of this. Regarding termination of $\to_{\mathcal{R}_F}^! \circ \to_{\mathcal{S}}$, we use Proposition 5 to obtain a simpler version $\mathcal{S}'$ of $\mathcal{S}$, namely, $\mathcal{S}'$ containing rules `A` and `B`. Termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ is equivalent to termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}'}$. The TRS $\mathcal{S}'$ is easily proved terminating. Since $\mathcal{R}_F \cup \mathcal{S}'$ is also terminating, we can conclude now that $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ is terminating. Hence, $\mathcal{R}$ is fairly-terminating w.r.t. $\mathcal{R}_F$.

7 Related work and conclusions

A number of other approaches to fairness within term rewriting have been developed so far. In particular, the notion of fairness as related to the removal of (residuals) of *redexes* rather than concerning the application of rules is well-known after O'Donnell's work [O'D77] on the so-called *outermost-fair reduction strategy* and the corresponding normalization results [O'D77, HL91]. O'Donnell's notion of fairness was intended to provide a basis for computing the normal form of terms. In those works, a (finite or infinite) reduction sequence $t_1 \to t_2 \to \cdots$ is fair if for all $i \geq 1$, and (position of a) redex $\varDelta$ in t_i, there is $j > i$ such that t_j does not contain any residual of $\varDelta$ [Ter03, Definition 4.9.10] (see also [Klo92]). It is not difficult to see that this notion of fairness is not comparable to ours.

Following these works, fairness plays a very important role in infinitary rewriting as an essential ingredient of strategies which intend to approximate infinitary normal forms [KKSV95]. The introduced notions, however, follow the previous style and become, then, uncomparable to ours.

Termination techniques have been recently proposed as suitable tools for proving *liveness properties of fair computations* [KZ05]. As in our approach, Koprowski and Zantema define fairness as relative to a given TRS. Their formal notion, however, is quite different: according to [KZ05, Sections 2.2 and 2.3], an infinite reduction in $\mathcal{R}_F \cup \mathcal{S}$ is called *fair* (w.r.t. $\mathcal{R}_F$) if it contains infinitely many $\mathcal{R}_F$-steps. No distinction between enabled and taken steps is made. This, of course, is a clear difference with the notion of fairness we are interested in. Moreover, the authors explicitly remark that *all fair reductions are infinite*. Thus, apart from the fact that this means that there are fair sequences in our sense which are not fair in Koprowski and Zantema's approach (e.g., the finite ones), no discussion about termination of such fair sequences is even possible!

In summary, we have shown that the problem of proving (rule) fair-termination of a TRS $\mathcal{R}$ w.r.t. a subTRS $\mathcal{R}_F$ can be reduced to the problem of proving termination of $\to^*_{\mathcal{S}} \circ \to_{\mathcal{R}_F}$ and $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ (where $\mathcal{S} = \mathcal{R} - \mathcal{R}_F$). We have proven that, if $\mathcal{R}_F$ is a single-rule TRS, fair-termination of $\mathcal{R}$ w.r.t. $\mathcal{R}_F$ is equivalent to termination of such relations. We have also investigated how to prove termination of $\to^!_{\mathcal{R}_F} \circ \to_{\mathcal{S}}$ as ordinary termination of TRSs. We can equivalently consider a subTRS $\mathcal{S}' \subseteq \mathcal{S}$ whose left-hand sides are $\mathcal{R}_F$-normal forms and then either prove termination of $\mathcal{R}_F \cup \mathcal{S}'$ (or even $\mathcal{S}'$ under some additional conditions), or transform $\mathcal{S}'$ into a TRS $\mathcal{U}_{\mathcal{S}'}$ and then prove *innermost* termination of $\mathcal{R}_F \cup \mathcal{U}_{\mathcal{S}'}$. Therefore, we always obtain (more) standard termination problems, namely: proving and disproving termination, innermost termination, and relative termination of TRSs, which can be addressed by existing termination tools. We believe that the results that we propose in this paper, although open to many extensions and generalizations, do indeed provide a quite practical *proof methodology* for proving fair-termination.

A number of interesting issues, however, remain to be investigated. For instance, Example 3 (which we cannot manage with our methodology) shows that a deeper analysis is needed to extend the use of termination techniques (and tools) for proving fair-termination. Regarding future extensions of our techniques, we think the following are interesting to consider:

1. The more general setting of *localized fairness* [Mes05] (also including weaker fairness notions like *justice* [Fra86, LPS81]).
2. The analysis of fair-termination *modulo* a set of equations; this notion has already been investigated by Porat and Francez [PF86].
3. Another important aspect of fairness is that, in many applications, only initial expressions satisfying concrete properties are expected to exhibit a fairly-terminating behavior. Indeed, this can be crucial to achieve fair termination in some cases.
4. The role of typing information in fair-termination. It is well-known that types play an important role in termination. As shown in [DLMMU04], it is possible to deal with termination of sorted TRS by reducing this problem to the problem of proving termination of a TRS (without sorts). We believe that a similar treatment could be useful for fair termination.

Of course, the implementation of our techniques in a system like MTT which is able to use external tools to solve termination problems is also envisaged (together with more experimentation on practical examples).

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