Polynomials over the reals in proofs of termination*

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Abstract. Polynomials over the real numbers were proposed as an alternative to polynomials over the naturals in termination proofs. We have recently shown how to use an arbitrary polynomial interpretation over the reals to generate well-founded and stable term orderings. Monotonicity can, then, be gradually introduced in the interpretations to deal with different applications. The first one is the generation of reduction orderings. We can also take advantage of non-fully monotonic polynomial interpretations in some remarkable cases. The dependency pairs method for proving termination of rewriting is an interesting one.

1 Introduction

Polynomials over the real numbers were proposed by Dershowitz [5] as an alternative to Lankford's polynomials over the naturals [6]. In contrast to Lankford's, well-foundedness has to be explicitly ensured by further requiring the subterm property; on the other hand, comparisons of terms by using the orderings induced by such polynomials are decidable.The automatic generation of such polynomials, however, has been hardly explored to date.

In [8], we have described how to associate a well-founded and stable ordering to an arbitrary polynomial interpretation over the reals, i.e., a collection $\{[f] \mid f \in \mathcal{F}\}$ of polynomials, where $[f]$ is a polynomial in $ar(f)$ variables whose coefficients are real numbers: $[f] \in \mathbb{R}[x_1, \ldots, x_{ar(f)}]$. For the purpose of this paper, we assume that $[f](x_1, \ldots, x_{ar(f)}) \geq 0$ for all $x_1, \ldots, x_{ar(f)}$ and $f \in \mathcal{F}$. Given a positive real number $\delta \in \mathbb{R}_{>0}$, a well-founded and stable (strict) ordering $>_\delta$ on terms is defined as follows: for all $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $t >_\delta s$ if and only if $[t] - [s] \geq_{\mathbb{R}} \delta$, where $[t]$ is the polynomial which is inductively obtained by interpreting each symbol f in t as $[f]$ and each variable $x \in \mathcal{V}ar(t)$ as a variable x ranging in $\mathbb{R}$. Monotonicity is ensured for each argument $i \in \{1, \ldots, ar(f)\}$ of each symbol $f \in \mathcal{F}$ if $\frac{\partial [f](x_1, \ldots, x_i, \ldots, x_{ar(f)})}{\partial x_i} \geq 1$.

Polynomial interpretations are well-suited to mechanize the proofs of termination. A proof of termination of a TRS is transformed into the problem of solving a set of constraints over the coefficients of a polynomial interpretation for the symbols of the TRS [6]. For practical reasons, we consider polynomials using nonnegative, *rational* coefficients. Thus, $[f] \in \mathbb{Q}_{\geq 0}[x_1, \ldots, x_k]$ for each k-ary symbol $f \in \mathcal{F}$. The tool MU-TERM[1] implements the previous approach to automatically generate μ-reduction orderings based on polynomial intepretations over the rationals. We use the C*i*ME system [4] to solve the set of constraints that we obtain. C*i*ME solves Diophantine inequations and yields non-negative integers as solutions. The use of rational numbers is easily made compatible with this limitation: given a Diophantine constraint $e_1 \geq e_2$ containing an occurrence of $\frac{p}{q}$ in e_1 (or e_2), we obtain an equivalent constraint $q \cdot e_1 \geq q \cdot e_2$. Then, we propagate the multiplication of q inside the members of e_1 and e_2 thus removing occurrences of q as a denominator. We repeat this process to remove all rational coefficients.

* Work partially supported by MCyT project TIC2001-2705-C03-01, MCyT Acción Integrada HU 2003-0003 and AVCyT grant GR03/025.

[1] See http://www.dsic.upv.es/~slucas/csr/termination/muterm.

The previous framework is well-suited for context-sensitive rewriting (*CSR* [7]). In *CSR*, a *replacement map* μ discriminates, for each symbol f of the signature, the argument positions $\mu(f)$ on which replacements are allowed. This can improve the termination behavior by pruning (all) infinite rewrite sequences[2]. Termination of *CSR* is fully captured by the so-called *μ-reduction orderings* [11], i.e., well-founded, stable orderings $>$ which are *μ-monotonic*, i.e., for all $f \in \mathcal{F}$ and $i \in \mu(f)$, $>$ is monotonic in the i-th argument of f. Term rewriting is a particular case of *CSR* where the replacement map $\mu_\top(f) = \{1, \ldots, ar(f)\}$, for all $f \in \mathcal{F}$ is used. Thus, polynomial $\mu_\top$-reduction orderings can also be used in proofs of termination. We will also see that more general μ-reduction orderings can also be useful in proofs of termination of rewriting.

2 Proofs of polynomial termination of TRSs

We do not know whether polynomial interpretations over the rationals (or reals) are actually more powerful than polynomial interpretations over the naturals. As far as the author knows, this is an open problem. In practice, however, they can be helpful:

Example 1. Consider the TRS $\mathcal{R}$:

```
a -> b          c -> d          b -> c
```

Most constraint solvers use some finite domain to give value to the unknowns. For instance, a system using a domain[3] $\{0, 1, 2\}$ would be unable to prove the termination of $\mathcal{R}$ by using polynomials over the naturals. However, the polynomial interpretation

```
[a] = 2        [b] = 1        [c] = 1/2        [d] = 0
```

(computed by MU-TERM for $\delta = \frac{1}{10}$) proves termination of $\mathcal{R}$.

Although this example looks somehow artificial, these chains of symbols naturally arise in some applications. For instance, proofs of termination of a Conditional TRS (CTRS) $\mathcal{R}$ are usually attempted by first transforming it into a TRS $\mathcal{U}(\mathcal{R})$ and then proving termination of $\mathcal{U}(\mathcal{R})$ (see [9] for an overview of these methods). In this setting, it is usual to introduce n new symbols $U_1, U_2, \ldots, U_n$ for each conditional rule $l \to r \Leftarrow s_1 = t_1, s_2 = t_2, \ldots, s_n = t_n$ which are related in the transformed system as in the previous example.

Example 2. Consider the following CTRS $\mathcal{R}$ [10, Example 3.4]:

```
f(x) -> g(y) <= x -> h(y), i(x) -> y        a -> b <= c -> d
i(x) -> a                                   c -> d
```

By using the transformation of [9, Definition 7.2.48], we get $\mathcal{U}(\mathcal{R})$:

```
f(X) -> u1(X,X)               i(X) -> a     a -> u(c)
u1(h(Y),X) -> u2(i(X),X,Y)    c -> d        u(d) -> b
u2(Y,X,Y) -> g(Y)
```

which can be proved terminating by the following polynomial interpretation (computed by MU-TERM with $\delta = \frac{1}{10}$):

```
[f](X) = 3.X + 1                     [h](X) = X + 3      [a] = 1        [b] = 0
[u1](X1,X2) = X1 + 2.X2              [i](X) = X + 3/2    [g](X) = X     [d] = 0
[u2](X1,X2,X3) = X1 + X2 + X3 + 1    [u](X) = X + 1/2    [c] = 1/3
```

No polynomial interpretation over the naturals using coefficients below 4 can directly prove termination of $\mathcal{U}(\mathcal{R})$.

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[2] See `http://www.dsic.upv.es/~slucas/csr/termination/examples`.

[3] This is the current default domain for C*i*ME and AProVE (see `http://www-i2.informatik.rwth-aachen.de/AProVE`).

3 Termination of TRSs using dependency pairs

A reduction pair $(\gtrsim, \sqsupset)$ consists of a reflexive, transitive, stable, and weakly monotonic relation $\gtrsim$ and a stable and well-founded ordering $\sqsupset$ satisfying either $\gtrsim \circ \sqsupset \subseteq \sqsupset$ or $\sqsupset \circ \gtrsim \subseteq \sqsupset$. Note that *monotonicity is not required* for $\sqsupset$. When using the dependency pairs method [1], we can prove termination by showing that the *lhs* and *rhs* of each rule of the TRS are comparable by using $\gtrsim$ whereas the components of each dependency pair are comparable by using $\sqsupset$ [2].

Example 3. Consider the TRS $\mathcal{R}$ which is part of the TPDB[4]:

```
f(f(X)) -> f(g(f(g(f(X)))))
f(g(f(X))) -> f(g(X))
```

Termination of $\mathcal{R}$ can be proved by finding a reduction pair $(\gtrsim, \sqsupset)$ such that:

`f(f(X))`	$\gtrsim$	`f(g(f(g(f(X)))))`	`F(f(X))`	$\sqsupset$	`F(g(f(g(f(X)))))`
`f(g(f(X)))`	$\gtrsim$	`f(g(X))`	`F(f(X))`	$\sqsupset$	`F(g(f(X)))`
			`F(f(X))`	$\sqsupset$	`F(X)`
			`F(g(f(X)))`	$\sqsupset$	`F(g(X))`

where `F` is introduced to define the dependency pairs (see [1]).

Given a polynomial interpretation over the reals, the relation $t \gtrsim s$ iff $[t] - [s] \geq 0$, is a quasi-ordering; this quasi-ordering is weakly monotonic in all arguments of all symbols provided that only nonnegative coefficients are used in the polynomials. Given a polynomial interpretation and $\delta > 0$, we have $\gtrsim \circ >_\delta \subseteq >_\delta$ (if there is u such that $[t] - [u] \geq 0$ and $[u] - [s] \geq \delta$, then $[t] - [u] + [u] - [s] = [t] - [s] \geq 0 + \delta = \delta$); thus $(\gtrsim, >_\delta)$ is a *reduction pair*. We have implemented the generation of such reduction pairs in MU-TERM: we just give a TRS $\mathcal{R}$ the *least* replacement map $\mu_\perp(f) = \emptyset$ for all $f \in \mathcal{F}$. Since $\mu_\perp$ expresses *no* monotonicity requirements for $>_\delta$, this is the most flexible choice we can do (although it is not the only one). MU-TERM (tries) to compute the interpretation which makes $\gtrsim$ and $>_\delta$ compatible with the rules and the dependency pairs as above.

Example 4. The following polynomial interpretation:

```
[f](X) = X + 1      [g](X) = 1/2.X      [nF_f](X) = X
```

(where `nF_f` is the MU-TERM representation of symbol `F` in Example 3) defines a reduction pair $(\gtrsim, >_\delta)$ (with $\delta = \frac{1}{10}$) which proves termination of $\mathcal{R}$ in Example 3.

In fact, Arts and Giesl already noticed that the polynomials used with dependency pairs *do not necessarily depend on all their arguments*.

Example 5. Consider the TRS $\mathcal{R}$ [1, Example 2]:

```
minus(X,0) -> X          minus(s(X),s(Y)) -> minus(X,Y)
quot(0,s(Y)) -> 0        quot(s(X),s(Y)) -> s(quot(minus(X,Y),s(Y)))
```

The following polynomial interpretation (computed by MU-TERM for $\delta = \frac{1}{10}$):

```
[minus](X1,X2) = X1     [s](X) = X + 1          [nF_minus](X1,X2) = X1
[0] = 0                 [quot](X1,X2) = X1      [nF_quot](X1,X2) = X1
```

defines a reduction pair $(\gtrsim, >_\delta)$ which proves termination of $\mathcal{R}$ in Example 3. The interpretation computed by MU-TERM exactly corresponds to the ad-hoc polynomial proof given by Arts and Giesl [1, page 142].

[4] Termination Problems Data Base, see `http://www.lsi.upc.es/~albert/tpdb.html` and also `http://www.lri.fr/~marche/wst2004-competition/tpdb/Rubio/aoto.trs`.

This nicely corresponds to polynomial μ-reduction orderings. Arts and Giesl, however, do not consider polynomials over the rationals in their proofs of termination. It is worth to note that the use of rational coefficients (between 0 and 1) to introduce non-monotonicity in the corresponding term orderings makes a difference which cannot be simulated by just using polynomials over the naturals (see [8, Section 4] for a deeper discussion in this respect). In the proof of termination of Example 3 above, the difference is noticeable in that the proof which uses polynomials over the rationals is pretty simple and automatic (Example 4), but it becomes difficult or impossible when more traditional base orderings are used in combination with the dependency pairs approach (e.g., RPOS or polynomials over the naturals).

4 Conclusion

The use of μ-reduction orderings based on polynomial interpretations over the real or rational numbers can play a role in proofs of termination of term rewriting. Moreover, we stress that μ-reduction orderings provide a more general framework and, in fact, other μ-reduction orderings (e.g., the context-sensitive recursive path ordering, CSRPO [3]) could also be suitable for implementing the necessary comparisons.

There are, however, many theoretical and practical issues that deserve further investigation. An exciting one is: *are the polynomial interpretations over the reals (or rationals) more powerful than polynomial interpretations over the naturals?* As remarked above, regarding the generation of term orderings which are *not* fully monotonic, the answer is yes (already for the rationals). Regarding the generation of monotonic term orderings, the answer is not clear yet. In particular, we do not know of any TRS which can be proved terminating by using a reduction ordering based on a polynomial interpretation over the reals (or rationals) but cannot be proved terminating by using a polynomial interpretation over the naturals.

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